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Division algebras over surfaces

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Abstract

We consider here a number of topics concerning the theory of division algebras over the function field of a surface. One result relates the obstruction for ramification data to be from a division algebra and third etale cohomology. Another result shows this obstruction is always zero when the surface is Spec of a regular local ring (with some mild assumptions). At the same time we study the Brauer group of this function field as it relates to the Brauer group of the function field of the henselization. Finally we prove a result which says that Brauer group elements which "look like" they are of prime index q (unequal to any characteristic) must have all their ramification split by a cyclic Galois extension of the same degree. This last result requires a primitive q root of one.

Keywords: Division algebra; Cyclic algebra; Ramification; Etale cohomology

Introduction

Let S be a nonsingular surface with field of fractions $K = F(S)$. By this we mean a two dimensional separated excellent integral Noetherian scheme quasi-projective over some affine scheme. Our goal in this paper is to prove several results about division algebras over K, or equivalently, about the Brauer group of K. One result is about algebras D/K of prime degree. We show that they are, in a weak sense, cyclic "up to ramification." That is, we show that there is a cyclic Galois extension of the same degree which splits all the ramification of D (7.13). Here we must assume K contains a needed root of unity and D has degree prime to any residue

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characteristic. In a separate direction, we explore the restrictions on ramification that a Brauer group element must have. We review and spell out the well known fact that the ramification of a Brauer group element must itself have ramification that sums to 0 at any point. But this obstruction is insufficient to completely describe the ramification of Brauer group elements. In Section 6 (6.12) we show that there is a further obstruction, which takes values in the kernel of $H^3(S, \mathcal{G}_m) \to H^3(F(S), \mathcal{G}_m)$ where the cohomology is etale and $F(S)$ is the field of fractions. This result can be, in the case of a nonsingular surface over a field, deduced from the result in e.g. [CHK]. Besides applying to more surfaces, the merit of Section 6 is the concrete way the material is developed. Note that in this section we make no assumptions about roots of unity, but do assume our Brauer group elements have order prime to any characteristic.

We would like to further understand this obstruction in third cohomology, and we prove one result about it here. Taking up much of this paper, is a proof of the following. Suppose $S = \mathrm{Spec}(R)$ for R a regular local two dimensional ring with field of fractions K. Then $H^3(S, \mathcal{G}_m) \to H^3(K, \mathcal{G}_m)$ in injective (5.2). That is, the obstruction above is always trivial for these S. As part of proving this last result, we realized the following surprising fact. Let R^h be the henselization of R with field of fractions K^h. Assume K has characteristic 2 or $K(\rho)/K$ is a cyclic Galois extension for ρ any 2^n root of one. Then $\mathrm{Br}(K) \to \mathrm{Br}(K^h)$ is surjective (5.1).

Let us review some notation and definitions. A curve on S will mean a codimension 1 integral subscheme and a point on S will be a codimension 2 integral subscheme. For a curve C or a point P, $F(C)$ or $F(P)$ will denote the corresponding residue field. For any torsion abelian group A, A' will refer to the prime to p part of the group, where p runs over the characteristics of all points. If R is a domain, then $q(R)$ is its field of fractions. If $L \supset K$ are fields, then $[L : K]$ is the degree of the extension. Suppose K is a field with characteristic prime to n which contains a primitive n root of one ρ. The symbol algebra $(x, y)_n$ is the central simple K algebra of degree n generated by ζ, γ, subject to the relations $\zeta^n = x$, $\gamma^n = y$, and $\zeta\gamma = \rho\gamma\zeta$. We also use $(x, y)_n$ to denote the corresponding class in the Brauer group $\mathrm{Br}(K)$.

For every curve $C \subset S$, the stalk $R_C = \mathcal{O}_{S,C}$ defines a discrete valuation domain and hence a ramification map $\mathrm{ram}_C : \mathrm{Br}(K)' \to H^1(F(C), \mathbb{Q}/\mathbb{Z})'$ that fits into the exact sequence: $0 \to \mathrm{Br}(R_C)' \to \mathrm{Br}(K)' \to H^1(F(C), \mathbb{Q}/\mathbb{Z})' \to 0$ where $H^1(F(C), \mathbb{Q}/\mathbb{Z}) = H^1(G_{F(C)}, \mathbb{Q}/\mathbb{Z})$ and $G_{F(C)}$ is the absolute Galois group of the residue field $F(C)$ (e.g. [LN, p. 68]). It will be convenient to write elements $\chi \in H^1(F(C), \mathbb{Q}/\mathbb{Z})$ in the following way. Note that $H^1(F(C), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_c(G_{F(C)}, \mathbb{Q}/\mathbb{Z})$, the subscript referring to continuous homomorphisms. Then χ has a kernel G_L where $L/F(C)$ is cyclic Galois of degree, say, n. χ is then determined by naming $\sigma \in \mathrm{Gal}(L/F(C))$ such that $\chi(\sigma) = 1/n + \mathbb{Z}$. For this reason we will identify χ with the pair $L/F(C), \sigma$.

We require two well known results about this ramification map, both of which involve its functoriality. To preserve generality, let R' be a discrete valuation domain with maximal ideal P' and with field of fractions K'. Let L'/K' be a finite field extension. Denote by S' the integral closure of R' in S' and P_i' the set of maximal ideals of S' which necessarily lie over P'. Set $k = R'/P'$ and $k_i = S'/P_i'$. The results we need detail how the ramification maps, ram, associated to R and ram_i, associated to $S'_{P_i'}$ behave with respect to the restriction map $\mathrm{Res} : \mathrm{Br}(K') \to \mathrm{Br}(L')$ and corestriction map $\mathrm{Cor} : \mathrm{Br}(L') \to \mathrm{Br}(K')$. We also have restriction maps $\mathrm{Res}_i : H^1(k, \mathbb{Q}/\mathbb{Z}) \to H^1(k_i, \mathbb{Q}/\mathbb{Z})$ and corestriction maps $\mathrm{Cor}_i : H^1(k_i, \mathbb{Q}/\mathbb{Z}) \to H^1(k, \mathbb{Q}/\mathbb{Z})$.

Proposition 0.1.

(a) *The following diagram commutes*:

$$\begin{array}{ccc} \mathrm{Br}(L')' & \stackrel{\mathrm{ram}_{S'}}{\longrightarrow} & \bigoplus_i H^1(k_i, \mathbb{Q}/\mathbb{Z}) \\ \mathrm{Res} \uparrow & & \uparrow \sum_i e_i \,\mathrm{Res}_i \\ \mathrm{Br}(K')' & \stackrel{\mathrm{ram}_R}{\longrightarrow} & H^1(k, \mathbb{Q}/\mathbb{Z}) \end{array}$$

where e_i is the ramification index of P_i' over P and $\mathrm{ram}_{S'}$ *is the direct sum of the ramification maps* ram_i.

(b) *The following diagram also commutes*:

$$\begin{array}{ccc} \mathrm{Br}(L')' & \stackrel{\mathrm{ram}_{S'}}{\longrightarrow} & \bigoplus_i H^1(k_i, \mathbb{Q}/\mathbb{Z}) \\ \mathrm{Cor} \downarrow & & \downarrow \sum_i \delta_i \,\mathrm{Cor}_i \\ \mathrm{Br}(K')' & \stackrel{\mathrm{ram}_R}{\longrightarrow} & H^1(k, \mathbb{Q}/\mathbb{Z}) \end{array}$$

where δ_i is the so-called defect of P_i'/P' defined as follows. Let L'' be such that L''/K' is separable and L'/L'' is purely inseparable. Let $L_i' \supset L_i''$ be the completions of $L' \supset L''$ at P_i'. Then the defect $\delta_i = [L' : L'']/[L_i' : L_i'']$.

Proof. Part (a) is in, for example, [LN, p. 69] and (b) is in [FSS, p. 921]. $\square$

We return to talking about the Brauer groups of surfaces and their fields of fractions. By [S1] we have that $\mathrm{Br}(S) = \bigcap_{C \subset S} \mathrm{Br}(\mathcal{O}_{S,C})$; the intersection being over all curves on S. Thus there is an exact sequence

$$0 \to \mathrm{Br}(S)' \to \mathrm{Br}(K)' \to \bigoplus_{C \subset S} H^1\big(F(C), \mathbb{Q}/\mathbb{Z}\big)'$$

and one goal of this paper is to study the cokernel.

Before we get to this, there are more facts we need to recall.

Lemma 0.2. *Suppose R is a ring and I, J are ideals of R. Then the following is exact*:

$$R \to R/I \oplus R/J \to R/(I + J)$$

where the second map is the difference of the canonical projections.

Proof. This is obviously a complex. Suppose $a, b \in R$ are such that $a + (I + J) = b + (I + J)$. Then $a - b = i + j$ where $i \in I$ and $j \in J$. Then $a - i = b + j$ is the preimage of $(a + I, b + J)$. $\square$

Next we need a few facts about regular rings.

Proposition 0.3. *Suppose R is a regular semilocal domain. Then*:

(a) *R is a UFD.*
(b) *Suppose $P_1, \dots, P_r$ are height one prime ideals such that R/P_i is regular and the $P_i + P_j/P_j$ are distinct prime ideals in R/P_j for all $i < j$. Then the sequence*

$$R \to \bigoplus_i R/P_i \to \bigoplus_{i \neq j} R/(P_i + P_j)$$

is exact.

Now suppose R is an excellent regular local ring of dimension 2 with residue field k. Suppose π, $\delta \in R$ are distinct primes and $\bar{R}_\pi$ is the integral closure of $R/\pi R$. Let $\tilde{\delta}$ be the image of δ in $R/\pi R \subset \bar{R}_\pi$.

(c) *$\bar{R}_\pi$ is semilocal and finite as a module over R.*

Let v_i be the discrete valuations associated to the maximal ideals of $\bar{R}_\pi$ and k_i the residue field associated to v_i. Let n be the length of $R/(\pi, \delta)$.

(d) *n is finite and equals $\sum_i v_i(\tilde{\delta})[k_i : k]$.*

Proof. Of course in the local case (a) is the famous result of Auslander–Buchsbaum (e.g. [Ma, p. 163]). If $P \subset R$ is a height one prime ideal, and $M \subset R$ is a maximal ideal, then PR_M is a principal ideal and hence free. It follows that P is projective, and hence principal since R is semilocal.

Turning to (b), we induct on r. If $r = 2$ then we are done by 0.2. If we assume the result for $r - 1$, then the proof for r follows from the claim that $P_1 \cap \cdots \cap P_{r-1} + P_r = (P_1 + P_r) \cap \cdots \cap (P_{r-1} + P_r)$. The inclusion left to right is clear. For the other direction, note that both sides contain P_r, so it suffices to prove these ideals have equal images in $\bar{R} = R/P_r$. For $a \in R$ let $\bar{a}$ be the image in $\bar{R}$. If π_i is the generator or P_i, then $P_1 \cap \cdots \cap P_{r-1}$ is $R\pi_1, \dots, \pi_{r-1}$. Thus the left side maps to $\bar{R}\bar{\pi}_1, \dots, \bar{\pi}_{r-1}$ which is equal to $\bar{R}\bar{\pi}_1 \cap \cdots \cap \bar{R}\bar{\pi}_{r-1}$ because $\bar{R}$ is a UFD. But this later expression is the image of the right side, proving (b).

Part (c) follows because of e.g. [Ma, p. 257]. Since (π, δ) must have height 2, n is finite and the equality can be quoted from [F, p. 412]. $\square$

We need two (closely related) very special cases of a result about henselian local rings from [EGA].

Proposition 0.4.

(a) *Suppose R is a henselian two dimensional regular local ring and $X \to \operatorname{Spec}(R)$ is the result of a sequence of point blow-ups with exceptional fiber $E = \bigcup_i E_i$. Assume $C \subset \operatorname{Spec}(R)$ is an irreducible curve with strict transform $C' \subset X$. Then C' intersects E in only one point.*
(b) *Suppose R is a henselian excellent local domain with integral closure $\bar{R}$. Then $\bar{R}$ is local.*

Proof. We begin with (a). The affine ring $\bar{R}$ of C is also henselian and we can set $\bar{X} \to C$ to be the pullback. The composition $f : C' \to \bar{X} \to C$ is proper and C' is connected. Thus by [EGA, p. 135] (18.5.19) the closed fiber of f has one point and this is the intersection of C' and E. As for (b), we use the same result applied to $\mathrm{Spec}(\bar{R}) \to \mathrm{Spec}(R)$ which is finite and hence proper. $\square$

1. Ramification of cyclic covers

In this section will describe and prove the basic result that the ramification of a Brauer group element must itself have trivial ramification when summed at any point. We begin by giving as concrete as possible description of the map which defines the ramification of a cyclic cover. This material is essentially standard [AM], but the published versions are less general, though not for any real reason.

In pursuit of the above goal, we develop machinery about the behavior of regular local rings under finite extensions. This is machinery we will use throughout this paper, and in this sense 1.2 is the most important result of this section.

Assume R is a regular excellent Noetherian local domain of dimension 2 with residue field k and maximal ideal $M = (x, y)$. Since M is the only maximal ideal, when we say $P \subset R$ is a prime ideal we will always mean height one prime. Suppose $P \subset R$ is a prime ideal and $q(R/P)$ is the field of fractions. Suppose $\chi \in H^1(q(R/P), \mathbb{Q}/\mathbb{Z})'$ is of order n and corresponds to $L/q(R/P), \sigma$. We assume n is prime to the characteristic of k. Let $\bar{R}_P$ be the integral closure of R/P in $q(R/P)$. Then $\bar{R}_P$ is semilocal with maximal ideals $M_1, \ldots, M_r$. Set $\bar{R}_{M_i}$ to be the localization of $\bar{R}_P$ at M_i, which is therefore a discrete valuation domain with valuation v_i and residue field k_i finite over k.

With respect to each v_i we have integers e_i, f_i, and g_i associated to $L/q(R/P)$ as follows. e_i is the ramification index and f_i is the residue degree of $L/q(R/P)$ with respect to v_i. The integer g_i is the number of valuations into which v_i splits in L. Thus $e_i f_i g_i = n$. The ramification (or inertial) subgroup of $\mathrm{Gal}(L/q(R/P))$ at v_i is generated by $\tau = \sigma^{f_i g_i}$. Let $\hat{K}_i$ be the completion of $q(R/P)$ with respect to v_i. We can choose an extension of v_i and use it to form the completed extension $\hat{L}_i$. Let $\hat{L}_i \supset \hat{K}'_i \supset \hat{K}_i$ be such that $\hat{K}'_i/\hat{K}_i$ is the maximum unramified subextension. We know $\hat{L}_i$ has the form $\hat{K}'_i(\pi'^{1/e_i})$ for a prime π' of $\hat{K}'_i$ and that the extension of τ generates the Galois group $\mathrm{Gal}(\hat{L}_i/\hat{K}'_i)$. Thus $\tau(\pi'^{1/e_i})/\pi'^{1/e_i}$ is a root of unity ρ_i of order e_i. To describe ρ_i without resorting to completions, let π_i be a prime of L defining our chosen extension of v_i. In $\hat{L}_i$, $\pi_i = \pi'^{1/e_i} u$ for u a unit. It follows that $\tau(\pi_i)/\pi_i$ maps to the root of unity ρ_i in the residue field of L with respect to π_i. But since n is prime to the characteristic, we can identify ρ_i with its image in the residue field. Also, σ must fix ρ_i. Since the extensions of v_i are conjugate, ρ_i is independent of the choice of extension. Finally, we can define our ramification map on $H^1(q(R/P), \mathbb{Q}/\mathbb{Z})$ by setting

$$r_P\big(L/q(R/P), \sigma\big) = \prod_i (\rho_i)^{[k_i:k]}.$$

Note that we write the operation in μ multiplicatively here while in all other abelian groups the operation is additive.

Having defined r_P for each P, we can define $r : \bigoplus_P H^1(q(R/P), \mathbb{Q}/\mathbb{Z}) \to \mu$ as the product of the r_P. One can compute that r preserves the Galois actions, if we give μ the dual Galois action. That is, if the range is given as μ^{-1}.

The gross restriction on ramification we mentioned above is simply the following result:

Theorem 1.1. *Let R be as above and $K = q(R)$. The composition* $\mathrm{Br}(K) \to \bigoplus_{P \subset R} H^1(q(R/P), \mathbb{Q}/\mathbb{Z}) \to \mu^{-1}$ *is the constant* 1 *map.*

Remark. Of course if S is a surface, 1.1 gives a restriction on the image of $\mathrm{Br}(F(S))$ in $\bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})$ for every closed point of S.

Proof. There is a proof in [S1] that avoids the Merkurjev–Suslin Theorem, but here we do not bother. However, we do need roots of one for this, so we delay the proof of 1.1 until we have studied the following kinds of extensions and their relation to the r_P maps.

Let $f(t) \in R[t]$ be a monic irreducible polynomial, and P a prime ideal of R. Let $\tilde{f}(t)$ be the image in $R/P[t]$, and $\bar{f}(t)$ the image in $k[t]$. As before, let $\bar{R}_P$ be the integral closure of R/P and let $\bar{R}_P$ have residue fields $k_1, \ldots, k_r$, corresponding to maximal ideals $M_1, \ldots, M_r$. All the k_i contain $k = R/M$ naturally. The degrees $[k_i : k]$ will be important to us. Finally, set $R' = R[t]/(f(t))$.

Proposition 1.2.

(a) *R' is semilocal. If $\bar{f}(t)$ is either separable or irreducible, R' is regular. If $\bar{f}(t)$ is irreducible, MR' is the unique maximal ideal of R'.*
(b) *The maximal ideals of R' all lie over M and are in one to one correspondence with the irreducible factors of $\bar{f}(t)$.*
(c) *The prime ideals P' of R' lying over P are in one to one correspondence with the irreducible factors of $\tilde{f}(t)$ as an element of $q(R/P)[t]$.*
(d) *Suppose the prime ideal P' over P corresponds to the factor $\tilde{h}(t)$ of $\tilde{f}(t)$ and the maximal ideal M' corresponds to the factor $\bar{g}(t)$ of $\bar{f}(t)$. Let $\bar{h}_i[t] \in k_i[t]$ be the image of $\tilde{h}(t)$. Then $P' \subset M'$ if and only if the following condition holds. For some i, $\bar{g}(t)$ and $\bar{h}_i(t)$ have a common factor.*

Proof. If $\bar{f}(t)$ is separable, R'/R is etale and so R' is regular. If $\bar{f}(t)$ is irreducible, $R'/MR' = k[t]/(\bar{f}(t))$ is a field and so R' is local with a 2 generator maximal ideal. This proves (a).

As for (b), consider $R'/MR' = R[t]/(M, f(t)) = k[t]/(\bar{f}(t))$. Since R'/R is integral, the maximal ideals of R' all lie over M and so correspond to maximal ideals of $R'/MR' = k[t]/(\bar{f}(t))$. Thus (b) is clear. Part (c) is similar, as follows. Let T be the localization of R' at $R - P$. Then the primes of R' over P correspond to the prime ideals of $T/PT = F(P)[t]/(\tilde{f}(t))$ and we proceed as in (b).

Turning to (d), note that $\bar{R}_P$ is a UFD and so $\tilde{h}(t)$ can be assumed to be monic with coefficients in $\bar{R}_P$. Thus the definition of $\bar{h}_i(t)$ makes sense. Let

$$X = k[t]/\big(\bar{g}(t)\big) \otimes_{(R/P)[t]} \bar{R}_P[t]/\big(\tilde{h}(t)\big)$$

where we recall that $k[t]/(\bar{g}(t)) = R'/M'$. Let $P'', M'' \subset (R/P)[t]$ be induced by P' and M' respectively, so $P'' = (\tilde{h}(t)) \cap (R/P)[t]$. Then $R'/P' = (R/P)[t]/P''$ injects into $\bar{R}_P[t]/(\tilde{h}(t))$ and the extension is integral. Clearly $M'X = 0 = P'X$ and $P' \subset M'$ if and only if $P' + M' \neq R'$. Thus if P' is not contained in M', $X = 0$. If $P' \subset M'$, then $P'' \subset M''$ and under the identification $R'/P' = R/P[t]/P''$, M' corresponds to M''. Thus X can be written,

$$X = R'/M' \otimes_{R'/P'} \bar{R}_P[t]/\big(\tilde{h}(t)\big)$$
$$= \big(\bar{R}_P[t]/\tilde{h}(t)\big)/M'\big(\bar{R}_P[t]/\tilde{h}(\tilde{t})\big)$$

and X is nonzero. Thus there is an induced surjection $\bar{R}_P[t]/(\tilde{h}(t)) \to X$ which factors through $(\bar{R}_P/M\bar{R}_P)[t]/(\bar{h}(t))$ where $\bar{h}(t)$ is the image of $\tilde{h}(t)$. Finally, $\bar{R}_P/M\bar{R}_P = \bigoplus_i S_i$ where each S_i is local with nilpotent maximal ideal and S_i has residue field k_i. Thus if X is nonzero, for some i, $B_i = k[t]/(\bar{g}(t)) \otimes_{k[t]} k_i[t]/(\bar{h}_i(t))$ is nonzero which is equivalent to $\bar{g}(t)$ and $\bar{h}_i(t)$ having a common factor. On the other hand, if B_i is nonzero the universal property of X shows there is a surjection $X \to B_i$ and so X is nonzero. Part (d) follows. $\square$

An extension of the form above, with $\bar{f}(t)$ either separable or irreducible, we call a **monic** extension. Of course if $\bar{f}(t)$ is separable respectively irreducible, respectively both, we call it a **separable**, **irreducible**, or **separable irreducible** monic extension.

We next consider separable monic extensions where $\bar{f}(t)$ has a divisor $t - \bar{a}$. Let $M' \subset R'$ be the maximal ideal associated to $t - \bar{a}$, and P a prime of R. Let M_i, $i = 1, \dots, r$, be the maximal ideals of $\bar{R}_P$ with residue fields k_i. Suppose P'_j is a prime ideal of R' corresponding to $\tilde{h}_j$, an irreducible factor of $\tilde{f}(t)$ over $\bar{R}_P$. Set $\bar{R}'_j = \bar{R}_P[t]/(\tilde{h}_j(t))$. Let $\bar{h}_{ji}(t)$ be the image of $\tilde{h}_j(t)$ in $k_i[t]$. Since all the $\bar{h}_{ji}(t)$ are separable, $\bar{R}'_j$ is etale over $\bar{R}_P$ and so integrally closed in its field of fractions. It follows that $\bar{R}'_j$ is the integral closure of $\bar{R}_P$. Arguing as in the proof of 1.2, the prime ideals of $\bar{R}'_j$ over M_i correspond to the irreducible factors of $\bar{h}_{ji}(t)$.

We next form $R'' = R'_{M'}$, the localization at M', and assume $P'_j \subset M'$. We can form R''/P'_jR'' which is the localization of R'/P'_j. Let $\bar{R}''_j$ be the integral closure of R''/P'_jR''. We have the diagram:

$$\begin{array}{ccccccc}
(R'/P'_j)_{M'} & = & R''/P'_jR'' & \subseteq & \bar{R}''_j & & \\
 & & \cup\vert & & \cup\vert & & \\
 & & R'/P'_j & \subseteq & \bar{R}'_j & = & \bar{R}_P[t]/(\tilde{h}_j(t)). \\
 & & \cup\vert & & \cup\vert & & \\
 & & R/P & \subseteq & \bar{R}_P & &
\end{array}$$

Since any localization of $\bar{R}'_j$ is integrally closed, it follows that in the above diagram $\bar{R}''_j$ is a localization of $\bar{R}'_j$. Since $P'_j \subset M'$, $\bar{a}$ is a root of $\bar{h}_{ji}(t)$ for some i. Since $t - \bar{a}$ has multiplicity one, for each i there is at most one j such that $\bar{a}$ is a root of $\bar{h}_{ji}(t)$. In terms of ideals, this says that for each M_i there is at most one prime P'_j over P and contained in M' such that M_i and $M'/P'_j \subset R'/P'_j$ are contained in a common maximal ideal M_{ji} of $\bar{R}'_j$.

Now fix an arbitrary M_i and let P'_j, for various j, be the full set of prime ideals over P where P'_j corresponds to $\tilde{h}_j(t)$. Since $\tilde{f}_j(t) = \prod_j \tilde{h}_j(t)$, $\bar{f}(t) = \prod_j \bar{h}_{ji}(t)$. Thus there is some j with $\bar{a}$ a root of $\bar{h}_{ji}(t)$. Translating again into ideals, this says that for each M_i there is one and only one prime ideal P'_j over P that is contained in M' and such that $\bar{R}''_j$ above has a maximal ideal

M_{ji} over M_i. Now the M_{ji} correspond to the factors of $\bar{h}_{ji}(t)$ with $\bar{a}$ as a root, and hence M_{ji} is unique. Furthermore, $R_P/M_i = \bar{R}''_j/M_{ji}$.

Of course, R'/R is finite etale, and so is R'/PR' as an extension of R/P. If we form $\bar{R}'_P = R'/PR' \otimes_{R/P} \bar{R}_P$, then $\bar{R}'_P/\bar{R}_P$ is finite etale. In particular, $\bar{R}'_P$ is integrally closed, implying that it is the direct sum of the $\bar{R}'_j$ indexed over the P'_j extending P in R'. Furthermore, R''/PR'' is a localization of R'/PR' and so $\bar{R}''_P = R''/PR'' \otimes_{R/P} \bar{R}_P$ is a localization of $\bar{R}'_P$ implying it is a direct sum of $\bar{R}''_j$, indexed over the P'_j in M'.

All together we have:

Corollary 1.3. *Suppose R' is a separable monic extension where the polynomial $\bar{f}(t)$ has factor $t - \bar{a}$ over k. Let R'', P, etc. be as above.*

(a) *For each maximal ideal M_i of $\bar{R}_P$, there is a unique height one prime ideal P''_j of R'', lying over P, such that there is some prime ideal M_{ji} of $\bar{R}''_j$ lying over M_i.*
(b) *The prime ideal M_{ji} above is unique, and is unramified over M_i with residue degree* 1. *These M_{ji} are all the prime ideals of $\bar{R}''_j$.*

Let R^h be the henselization of R. By e.g. [M, p. 36] we have that R^h is a direct limit of extensions of the form R'' above. Suppose P is generated by η and η factors into primes $\eta_1, \ldots, \eta_s$ in R^h, so the η_i exactly generate the prime ideals P_i of R^h over P. Let $\bar{R}^h_{P_i}$ be the integral closure of R^h/P_i which we know is local with maximal ideal we write as M^h_i. Since R^h is the direct limit of the R'' above, we can use the above to show:

Corollary 1.4.

(a) *$\bar{R}_P \subset \bar{R}^h_{P_i}$ and we can set $M_i = M^h_i \cap \bar{R}_P$. Then M_i is a maximal ideal and $M_i \neq M_j$ if $i \neq j$. The M_i are all the maximal ideals of $\bar{R}_P$. Furthermore, $\bar{R}_P/M_i = \bar{R}^h_{P_i}/M^h_i$.*
(b) *Suppose $R \supset P$, $f(t)$, and R'' are as in* 1.3. *Let $\chi \in H^1(q(R/P), \mathbb{Q}/\mathbb{Z})$ have image $\chi_j \in H^1(q(R''/P''_j), \mathbb{Q}/\mathbb{Z})$ for each prime ideal P''_j of R'' over P. Then*:

$$r_P(\chi) = \prod_j r_{P''_j}(\chi_j).$$

(c) *If, instead, $R'' = R^h$ is the henselization of R and P_j are the prime ideals of R_h over P,*

$$r_P(\chi) = \prod_j r_{P_j}(\chi_j).$$

Proof. Part (a) is easy using 1.3 and taking direct limits. To finish (b) note that R'' has the same residue field k as R and so the degrees of $\bar{R}_P/M_i$ over k and $\bar{R}''/M_{ji}$ over k are the same. As for (c), note that the henselization is a direct limit of extensions of the form treated in (a). □

Given 1.4, it is convenient to now assume R is henselian. Thus R/P is henselian for any prime ideal P, and $\bar{R}_P$ has a unique maximal ideal we denote by $M = (\pi)$ with residue field k_P. We next consider separable irreducible monic extensions. That is, we set $R' = R[t]/(f(t))$ where

$\bar{f}(t)$ is separable irreducible. Then R' is regular local henselian and we let P'_j be the prime ideals of R' over P. Set k' to be the residue field of R'. Since R'/R is finite etale, PR' is the intersection of the P'_j. Now the P'_j correspond to irreducible factors of $\tilde{f}(t) \in \bar{R}_P[t]$ and if $\tilde{h}_j(t)$ is one such factor, define $\bar{R}'_j = \bar{R}_P[t]/(\tilde{h}_j(t))$. It follows that $\bar{R}'_j$ is integrally closed and is the integral closure of R'/P'_j. By the henselian assumption $\bar{R}'_j$ has a unique prime ideal M'_j over M and the residue degree is n_j, the degree of $\tilde{h}_j(t)$ which is also the degree of $\bar{R}'_j$ over $\bar{R}_P$. If k'_j is the residue field of M'_j, then $[k'_j : k] = n_j[k_P : k] = [k'_j : k']n$. Thus $\sum_j [k'_j : k'] = [k_P : k]$ and we have:

Corollary 1.5.

(a) *Suppose R is henselian and R' is a separable irreducible monic extension. If $\alpha \in H^1(q(R/P), \mathbb{Q}/\mathbb{Z})$ and $P'_j \subset R'$ are the extensions of P, then*

$$r_P(\alpha) = \prod_j r_{P'_j}(\alpha_j)$$

where α_j is the image of α in $H^1(q(R'/P'_j), \mathbb{Q}/\mathbb{Z})$.

(b) *If R is a local regular 2 dimensional excellent ring and R' is the strict henselization of R, then the same formula holds.*

Proof. As before, part (a) is clear. As for (b), by 1.4 we can assume R is henselian. The strict henselization is the direct limit of extensions as in (a). □

We are ready to prove 1.1. By 1.4 and 1.5 it suffices to prove this when R is strictly henselian. In this case R has all needed roots of one and we can apply the Merkurjev–Suslin Theorem. That is, it suffices to prove 1.1 for any Brauer group element which is the class of a symbol algebra $(a, b)_n$ where a, b are prime elements of R. The result 1.1 now follows from 0.3(d).

In arguments to come we will want k_P/k to be purely inseparable and we use separable irreducible monic extensions to achieve this. To this end, suppose $\bar{a} \in k_P$ but $\bar{a} \notin k$. Let $\bar{f}(t) \in k[t]$ be the minimal polynomial of a which we assume is separable of degree n. Let $f(t) \in R[t]$ be a monic preimage as in 1.5 and $\tilde{f}$ its image in $R/P[t] \subset \bar{R}_P[t]$. Then by the henselian property, $\tilde{f}(t)$ has a linear factor $t - \tilde{a} \in \bar{R}_P[t]$ and we can let $P' \subset R' = R[t]/(f(t))$ be the corresponding prime ideal. It follows that $q(R/P) = q(R'/P')$ and $\bar{R}_P$ is also the integral closure of R'/P'. If $\alpha \in H^1(q(R/P), \mathbb{Q}/\mathbb{Z}) = H^1(q(R'/P'), \mathbb{Q}/\mathbb{Z})$ then the formula makes clear that $nr_{P'}(\alpha) = r_P(\alpha)$.

We are ready to state the following result, whose proof is clear.

Theorem 1.6. *Let R be as above, including that R is henselian (but not necessarily strictly henselian). Then there is a separable irreducible monic extension R'/R such that the following holds. First, P has an extension P' in R' such that $q(R/P) = q(R'/P')$. Second, if k' is the residue field of R' and $k'_{P'}$ is the residue field of the integral closure of R'/P', then $k'_{P'}/k'$ is purely inseparable. Finally, if $\alpha \in H^1(q(R/P), \mathbb{Q}/\mathbb{Z}) = H^1(q(R'/P'), \mathbb{Q}/\mathbb{Z})$ then $r_{P'}(\alpha)^{[k':k]} = r_P(\alpha)$.*

2. A result on regular rings

We want to prove a result on the Brauer group of $q(R)$ where R is a regular local ring of dimension two. To this end, we will have to blow-up to resolve the singularities of curves on $\mathrm{Spec}(R)$. The first step is the observation of this section, which concerns the class group of these blow-ups. This material is well known, and we include it for the ease of the reader.

Let R be a two dimensional Noetherian excellent regular local ring with maximal ideal M and residue field $k = R/M$. Let $\pi \in R$ be a prime element. We say π is nonsingular if the curve $\pi = 0$ goes through the point nonsingularly, that is, if $\pi \in M - M^2$. If δ is a singular prime of R, we know by [L] (see also [A]) that we can resolve this singularity.

Before stating the result of this section precisely, let us recall some elementary facts to fix notation. If S is a regular surface and $P \in S$ a (closed) point, we can define the blow-up $f : S' \to S$ such that f is proper, the fiber of P is the exceptional line E and this map induces an isomorphism $S' - E \cong S - P$. If $C \subset S$ is an irreducible curve containing P, then the closure of $f^{-1}(C - P)$ is an irreducible curve $C' \subset S'$ called the strict transform of C. If C contains P nonsingularly, note that $C' \cong C$. If C does not contain P, we also call $C' \cong f^{-1}(C)$ the strict transform. On the level of divisors, $f^*(C) = C' + nE$ where n is the multiplicity of P on C. Then it is easy to see that the induced map $\mathrm{Pic}(S) \to \mathrm{Pic}(S')$ is injective and has cokernel freely generated by the divisor class of E.

The singularity of δ will be resolved by a sequence of point blow-ups $X_n \to \cdots \to X_0 = \mathrm{Spec}(R)$. If E_i is the exceptional divisor of $X_i \to X_{i-1}$, we identify E_i with its strict transform in any X_j for $j > i$. It is then clear that the fiber of $X_n \to \mathrm{Spec}(R)$ over the unique point P is a tree of E_i's. Since R is a UFD, $\mathrm{Pic}(X_n) \cong \mathbb{Z}^n$ has as basis the divisor classes of the E_i. Finally, X_n resolves the singularity of δ if the strict transform, C_n, of $\delta = 0$ is nonsingular and the inverse image of the reduced subscheme $\delta = 0$ has normal crossings. In our situation, this means that the finitely many intersection points of C_n on $\bigcup E_i$ are all nonsingular points, and are each on a single E_i. Furthermore, if C_n and E_i meet at P, they do so with distinct tangents.

Now consider R/δ. This is a one dimensional domain. Let $\bar{R}_\delta$ be the integral closure of R/δ in its field of fractions $F(\delta)$. By our assumptions on R, $\bar{R}_\delta$ is finite over R/δ and hence is a semilocal PID with maximal ideals we label as $M_1, \ldots, M_r$. Let $\bar{R}_i$ be the localization of $\bar{R}_\delta$ at M_i, which is therefore a discrete valuation domain.

Theorem 2.1. *Suppose $X \to \mathrm{Spec}(R)$ resolves the curve defined by $\delta = 0$. Let $C \subset X$ be the strict transform of $\delta = 0$ which is nonsingular. Let P_j be the closed points of C which are the intersection points of C and the exceptional divisor. Let $R_j = \mathcal{O}_{X,P_j}$ be the stalk of X at P_j and $\delta_j \in R_j$ a prime locally defining C at P_j. Then $C = \mathrm{Spec}(\bar{R}_\delta)$. There is a one to one correspondence between the M_i and the P_j such that $\bar{R}_i$ is R_j.*

Proof. The induced $f : C \to \{\delta = 0\}$ is projective, quasifinite and therefore finite [H, p. 280]. Thus $C = \mathrm{Spec}(T)$ is affine and finite over R/δ. Since C is nonsingular, T is integrally closed. Since f is birational, $T = \bar{R}_\delta$. Any closed point of C must lie over P and hence is on the intersection of C and the exceptional divisor. □

We next want a theorem making very concrete the Picard group of a sequence of blow-ups of $X \to \mathrm{Spec}(R)$. If $X \to X'$ is the last blow-up in the sequence, at a point $P' \in X'$, let E be the exceptional line. If $E' \subset X'$ is the exceptional line of a previous blow-up in the sequence, we identify E' with its strict transform in X. The $E_1, \ldots, E_n \subset X$ can be defined as the exceptional lines of all the blow-ups. Every closed point is on one of the E_i, and is at most on 2 of them. If C is the sum (including difference) of strict transforms of curves on $\mathrm{Spec}(R)$, we call C an **ordinary** divisor. Of course, every element in the group of divisors of X is the sum of E_i and an ordinary divisor, as the next theorem makes explicit. However, before we state this result we need to explain a bit about intersection multiplicity which we use in the result.

Let S be a regular surface, and C, D distinct irreducible reduced curves on S containing the closed point P. In the stalk $R = \mathcal{O}_{S,P}$ these curves are principal defined by $\pi = 0$ and $\delta = 0$ and $(\pi), (\delta) \subset R$ are distinct prime ideals. Then (π, δ) is a height 2 ideal and of course the intersection multiplicity of C and D at P, written $(C \cdot D)_P$, is the length of $R/(\pi, \delta)$. Suppose k is a field with $k(P)/k$ finite for all closed points P. Then we define $(C \cdot D)_k = \sum_{P \in D} (C \cdot D)_P [k' : k]$. In every case, the D (and sometimes C) will be $\mathbb{P}^1_k$. Note that as long as we avoid self intersections, this definition can be extended to divisors. Of course we have:

Lemma 2.2. *Suppose S' is a regular surface and $C' \subset S'$ is a curve containing a closed point P. Let $\phi : S \to S'$ be the blow-up at P and $E \subset S$ the exceptional line $\mathbb{P}^1_k$ where $k = k(P)$. Let $C \subset S$ be the strict transform of C'. Then $(C \cdot E)_k$ is the multiplicity of C' through P. In particular, $\phi^*(C') = C + (C \cdot E)_k E$ as divisors.*

Proof. Since this question is local at P we can assume $S' = \mathrm{Spec}(R)$ where R is a regular local two dimensional ring. Then C' is defined by $\pi \in R$. If M is the maximal ideal of R, then $\pi \in M^m - M^{m+1}$ where m is the multiplicity. If $M = (x, y)$, then M^m/M^{m+1} can be identified with homogeneous polynomials of degree m in the images $\bar{x}, \bar{y}$ over $k = R/M$. If $\bar{\pi} \in M^m/M^{m+1}$ is the image of π, then the zeroes of $\bar{\pi}$ on $E = \mathbb{P}^1_k$ are the intersection points of $C \cap E$ and the result is clear. $\square$

Theorem 2.3. *Let X be as in* 2.1 *and $k = R/M$. There are positive integers $a_{ij} \in \mathbb{Z}$ such that if $C \subset X$ is an ordinary divisor, then*

$$C + \sum_{ij} a_{ij} (C \cdot E_i)_k E_j = 0$$

in $\mathrm{Pic}(X)$.

Proof. We proceed by induction on n, i.e., by induction on the number of blow-ups. If $X = \mathrm{Spec}(R)$, the result is trivial. Suppose $\pi : X \to X'$ is the blow-up of $P' \in X'$ creating $E_n \subset X$ as the exceptional line, and $E'_1, \ldots, E'_{n-1}$ are the exceptional lines in X'. Set $k' = F(P')$. Assume the result for X' and let $C \subset X$ be an ordinary divisor with image $C' \subset X'$. Then there are positive integers a'_{ij} such that $C' + \sum_{i,j=1}^{n-1} a'_{ij} (C' \cdot E'_i)_k E'_j = 0$. Let $E_i \subset X$ be the strict transforms of E'_i. Then $\pi^*(C') = C + (C \cdot E_n)_{k'} E_n$ and $\pi^*(E'_i) = E_i + (E_i \cdot E_n)_{k'} E_n$. As mentioned, we know $(E_i \cdot E_n)_{k'} = 0$ except it equals 1 for either one or two i, which we can assume are $i = 1$ (case 1) or $i = 1, 2$ (case 2). We have:

$$0 = C + (C \cdot E_n)_k [k' : k] E_n + \sum_{i,j=1}^{n-1} a'_{ij} \big(C' \cdot E'_i\big)_k E_j + \left(\sum_{i,j=1}^{n-1} a'_{ij} \big(C' \cdot E'_i\big)_k (E_j \cdot E_n)_k [k' : k] \right) E_n.$$

We note that by [F, p. 142], $(C' \cdot E'_i)_k = (C \cdot E_i)_k$ if $P' \notin E'_i$ and $(C' \cdot E'_i)_k = (C \cdot E_i)_k + (C \cdot E_n)_k[k':k]$ if not. In other words, $(C' \cdot E'_i)_k = (C \cdot E_i)_k + (C \cdot E_n)_k(E_i \cdot E_n)_k$. Substituting and rearranging we have:

$$\begin{aligned}
0 &= C + \left[(C \cdot E_n)_k[k':k] + \sum_{i,j=1}^{n-1} a'_{ij}\big((C \cdot E_i)_k + (C \cdot E_n)_k(E_i \cdot E_n)_k\big)(E_j \cdot E_n)_k[k':k]\right]E_n \\
&\quad + \sum_{i,j=1}^{n-1} a'_{ij}\big((C \cdot E_i)_k + (C \cdot E_n)_k\big)E_j \\
&= C + \left[(C \cdot E_n)_k[k':k] + \sum_{i,j=1}^{n-1} a'_{ij}(C \cdot E_n)_k(E_i \cdot E_n)_k(E_j \cdot E_n)_k[k':k]\right]E_n \\
&\quad \sum_{i,j=1}^{n-1} a'_{ij}(C \cdot E_i)_k(E_j \cdot E_n)_k[k':k]E_n + \sum_{i,j=1}^{n-1} a'_{ij}(C \cdot E_i)_k E_j \\
&\quad + \sum_{i,j=1}^{n-1} a'_{ij}(C \cdot E_n)_k(E_i \cdot E_n)_k E_j.
\end{aligned}$$

In case 2, if we set

$$\begin{gathered}
a_{nn} = [k':k] + \big(a'_{11} + a'_{12} + a'_{21} + a'_{22}\big)[k':k]^3, \\
a_{in} = \big(a'_{i1} + a'_{i2}\big)[k':k], \qquad a_{nj} = \big(a'_{1j} + a'_{2j}\big)[k':k]
\end{gathered}$$

and in all other cases

$$a_{ij} = a'_{ij}$$

then the theorem holds. Similarly, in case 1 we set

$$\begin{gathered}
a_{nn} = [k':k] + a'_{11}[k':k]^3, \\
a_{in} = a'_{i1}[k':k], \qquad a_{nj} = a'_{1j}[k':k]
\end{gathered}$$

and in all other cases

$$a_{ij} = a'_{ij}$$

and the theorem also holds. $\square$

3. Unipoint curves

Let R be as usual, a regular local ring of dimension 2. In Section 6 we will prove a result about the relationship between the Brauer group of $q(R)$ and $q(R^h)$ where R^h is the henselization. To accomplish this we will need to show there are plenty of prime ideals in R which are well behaved (do not split) when we go to R^h. This is the goal of this section.

Let $X \to \operatorname{Spec}(R)$ be a sequence of point blow-ups at points and k the residue field of R. Let $E = \bigcup E_i \subset X$ be the union of all the exceptional lines. Any other curve is the strict transform of a curve on $\operatorname{Spec}(R)$, and are called **ordinary** curves. We say an ordinary irreducible curve $C \subset X$ is **unipoint** with respect to X if $C \cap E$ is a single point, necessarily the only point on C, and C is nonsingular at that point. We say $C' \subset \operatorname{Spec}(R)$ is unipoint if its strict transform is unipoint.

It will be necessary to talk about unipoint curves without reference to X. To this end we claim:

Theorem 3.1. *Suppose $X \to \operatorname{Spec}(R)$ and $X' \to \operatorname{Spec}(R)$ are sequences of point blow-ups. Assume $C \subset \operatorname{Spec}(R)$ is unipoint with respect to X. Then the strict transform of C in X' intersects the exceptional divisor in one point.*

Proof. In order to accomplish this we observe:

Lemma 3.2.

(a) *Suppose $X'_1 \to X_1$ is a sequence of point blow-ups and $P_1 \in X_1$ is a point whose inverse image is a point P'_1 in X'_1. Let $X'_2 \to X'_1$ be the blow-up at P'_1. Then $X'_2 \to X_1$ factors as $X'_2 \to X_2 \to X_1$ where $X_2 \to X_1$ is the blow-up at P_1 and $X'_2 \to X_2$ is a sequence of point blow-ups.*

(b) *In the situation of* 3.1, *there is a sequence of point blow-ups $X'' \to X$ such that $X'' \to \operatorname{Spec}(R)$ factors through X' and $X'' \to X'$ is also a sequence of point blow-ups.*

Proof. To begin with (a), we are claiming that there is the following diagram:

$$\begin{array}{ccccc} X'_2 = Bl_{P'_1}(X'_1) & \longrightarrow & X'_1 & \longrightarrow & X_1 \\ || & & & & || \\ X'_2 & \longrightarrow & X_2 = Bl_{P_1}(X_1) & \longrightarrow & X_1 \end{array}$$

where the bottom row needs to be proven and $Bl_P(X)$ is the blow-up at P. There is an open $U_1 \subset X_1$ such that $P_1 \in U_1$ and $X'_1 \to X_1$ is an isomorphism on U_1. There is an open $U_2 \subset X_1$ such that $U_1 \cup U_2 = X_1$, and $P_1 \notin U_2$. Let $U'_2 \subset X'_1$ be the inverse image. Then X'_1 is the union of U_1 and U'_2 patched along $U'_2 \cap U_1 \cong U_2 \cap U_1$.

If $X_2 \to X_1$ is the blow-up at P_1, then this is an isomorphism on U_2 and we can let $U'_1 \subset X_2$ be the inverse image of U_1. X_2 is the union of U'_1 and U_2 patched along $U'_1 \cap U_2 \cong U_1 \cap U_2$. Thus X'_2 can be defined by patching U'_2 and U'_1 along $U'_2 \cap U'_1 \cong U_1 \cap U_2$ and (a) is clear.

We turn to proving (b). We accomplish this by inducting on the number of blow-ups in X'. Using induction, we assume the following. There is a sequence of point blow-ups $X_1 \to \operatorname{Spec}(R)$ such that $X' \to X'_1$ is the blow-up at $P'_1 \in X'_1$. Furthermore, there is a sequence of point blow-ups $X_1 \to X_1'$ such that the composition $X_1 \to \mathrm{Spec}(R)$ factors as $X_1 \to X \to \mathrm{Spec}(R)$ and $X_1' \to X$ is a sequence of point blow-ups. The picture is:

$$\begin{array}{ccc} X' & & \\ \downarrow & & \\ X_1' & \leftarrow & X_1 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \leftarrow & X. \end{array}$$

There are two cases. If P_1' has inverse image a point $P_1 \in X_1$, then we let $X'' \to X_1$ be the blow-up at P_1 are we are done by (a). If the inverse image of P_1' is not a point, then $X_1 \to X_1'$ factors into $X_1 \to X_2 \to X_2' \to X_1'$ where the following holds. First, that P_1' has inverse image a point $P_2' \in X_2'$ and $X_2 \to X_2'$ is the blow-up at P_2'. By (a), $X_2 \to X_1'$ factors as $X_2 \to X' \to X_1'$ where $X_2 \to X'$ is a sequence of point blow-ups. We now set $X'' = X_1$ and the map $X'' \to X'$ is the composition $X_1 \to X_2 \to X'$. $\square$

We now turn to the proof of 3.1. Let $C_1 \subset X$ be the strict transform of C, which by definition intersects the exceptional divisor nonsingularly in one point. Let $X'' \to X$ be as in 3.2(b). Then the strict transform, $C_2 \subset X''$, of C_1, intersects the exceptional divisor in one point. In particular, C_2 has a unique closed point. If $C' \subset X'$ is the strict transform of C, then $C_2 \to C'$ is surjective and we are done. $\square$

By combining 3.1, 2.1, and 1.4(a) we have:

Corollary 3.3. *Suppose R is a regular local two dimensional domain with henselization R^h. If $\pi \in R$ is a unipoint prime, then π is prime in R^h.*

If $C \subset \mathrm{Spec}(R)$ corresponds to a unipoint prime π, we say C is a unipoint curve. We next prove the existence of many unipoint curves. As above, let $X \to \mathrm{Spec}(R)$ be a succession of blow-ups. Let $E_1, \ldots, E_n$ represent the exceptional lines.

Theorem 3.4. *For every point $P \in X$ there is a unipoint curve C containing P and nonsingular at P.*

Proof. First we consider the trivial case R is henselian. Let R' be the stalk of X at P. Choose $\pi' \in R'$ a prime nonsingular at P such that $(\pi') \subset \mathrm{Spec}(R')$ is not an exceptional divisor. Let $C \subset \mathrm{Spec}(R)$ be the image of (π'). Said differently, we know from 2.3 that there is a π' such that its only ordinary component C' is the closure of the curve of π' in $\mathrm{Spec}(R')$. Let $C \subset \mathrm{Spec}(R)$ be the image of C'. By 0.4, C is unipoint.

Of course we now have to handle the general case. As a first step, notice that two principal divisors have the same intersections with the E_i if they are "close."

Theorem 3.5. *Suppose R is a 2 dimensional regular local ring with maximal ideal M and $X \to \mathrm{Spec}(R)$ is a sequence of blow-ups as above. Let $\pi \in R$. There is an N such that if $\pi' - \pi \in M^N$, then (π) and (π') intersect the exceptional lines in the same points with the same multiplicities.*

It clearly suffices to show the following:

Proposition 3.6. *Suppose $X \to X'$ is the blow-up of P' and $R' = \mathcal{O}_{X',P'}$ with maximal ideal M'. Let n be an integer and π as above. Then there is an integer n' such that the following holds. Let $P \in X$ be a point on the preimage E of P' and $R_P = \mathcal{O}_{X,P}$ with maximal ideal M_P. Then if $\pi' \in R'$ satisfies $\pi - \pi' \in M'^{n'}$, π and π' have the same multiplicity m at P'. Furthermore, the strict transform of (π) and (π') intersect E at the same points with the same multiplicities. Finally, if P is one of those points, and $s(\pi) \in R_P$, $s(\pi') \in R_P$ are elements defining these strict transforms at P, we also have that $s(\pi) - s(\pi') \in M_P^n$.*

Proof. We assume $\pi - \pi' \in M'^{n'}$ with the n' to be chosen as we proceed. Let m be the multiplicity of π at P'. Then if $n' > m$, it is clear that π' has the same multiplicity m. By repeating the argument once, it suffices to prove the rest of this proposition for the affine blow-up $S \supset R'$ where $P \in \mathrm{Spec}(S)$. There are $x, y \in R'$ generating M' such that $S = R'[x/y]$. Furthermore, on $\mathrm{Spec}(S)$ the strict transform of (π) and (π') are defined by $s(\pi) = y^{-m}\pi$ and $s(\pi') = y^{-m}\pi'$.

Write $\pi = \sum_i a_i x^i y^{m-i}$ where some $a_i \in R'^*$. Since M^m/M^{m+1} has basis the images of $x^i y^{m-i}$, it follows that the i where $a_i \in R'^*$ and the image of such a_i in R'/M' are uniquely determined by knowing π modulo M^{m+1}. In other words, $\pi' = \sum_i a'_i x^i y^{m-i}$ where $a_i - a'_i \in M'$. $S/M'S = R'/M'[\bar{z}]$ where $\bar{z}$ is the image of x/y. In other words, the intersection points of $s(\pi)$ on $\mathrm{Spec}(S) \cap E$ correspond to the factors of $\sum_i \bar{a}_i z^i = \sum_i \bar{a}'_i z^i$ and so $s(\pi)$ and $s(\pi')$ have the same such points with the same multiplicities.

Finally suppose $n' = m + n$. Write $\pi - \pi' = \sum b_j x^j y^{m+n-j}$. Then $s(\pi) - s(\pi') = y^{-m} \sum_j b_j x^j y^{m+n-j} = \sum_j b_j (x/y)^j y^n \in M'^n S$. Since $M_P^n \supset M'^n S$, we are done. □

Now we can finish the proof of 3.4 in the general case. Let R be as in 3.4 with completion $\hat{R}$. Of course, $\hat{R}$ is henselian. Let $X \to \mathrm{Spec}(R)$ be a sequence of blow-ups at rational points with exceptional locus $E = \bigcup_i E_i$. Set $\hat{X} = X \times_R \mathrm{Spec}(\hat{R})$. Clearly $\hat{X} \to \mathrm{Spec}(\hat{R})$ is an induced sequence of blow-ups with exceptional locus isomorphic to E. Let $P \in E$ define $\hat{P} \in \hat{X}$. By the henselian case of 3.4, there is a unipoint curve $\hat{C} \subset \mathrm{Spec}(\hat{R})$ such that its strict transform $\hat{C}_1 \subset \hat{X}$ contains $\hat{P}$ and is nonsingular at $\hat{P}$. Let the prime $\hat{\pi} \in \hat{R}$ define $\hat{C}$. Let N be the integer whose existence is guaranteed by 3.5. There is a $\pi \in R$ such that $\hat{\pi} - \pi \in \hat{M}^N$. Consider the curve $C_2 \subset \mathrm{Spec}(R)$ defined by π and $\hat{C}_2 = C_2 \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\hat{R})$ the curve on $\mathrm{Spec}(\hat{R})$ defined by π. Finally, let $C_3 \subset X$ be the strict transform of C_2 and $\hat{C}_3 = C_3 \times_X \hat{X}$ the strict transform of $\hat{C}_2$. By 3.5 $\hat{C}_3$ and $\hat{C}$ intersect E in the same points implying that $\hat{C}_3$ is a unipoint curve. But it follows that C_3 is a unipoint curve as needed to prove 3.4. □

Next we use these unipoint curves to find elements. The situation is that we have $X \to \mathrm{Spec}(R)$ which is a sequence of blow-ups. If $E \subset X$ is an exceptional line, we call E a **limb** if E intersects the rest of the exceptional lines in only one point. Before we state the result we require an easy observation about $\mathbb{P}^1$.

Lemma 3.7. *Let P', Q' be distinct k'' rational points on $\mathbb{P}^1_{k''}$, and assume k'' has more than three elements. Let $c \in k''^*$. Then there is a rational function $\alpha \in k''(\mathbb{P}^1)$ such that α has exactly one simple zero and one simple pole, both rational points, and α has value 1 at Q' and value c at P'. Any other rational function with the same simple zero and pole is a k''^* multiple of α.*

Proof. The last sentence is obvious. As for the rest, by looking at an affine piece we know α has the form $d(t-u)/(t-v)$ for $d \in k''^*$ and $u, v \in k''$. Let P', Q' correspond to $t = x$ and $t = y$. Since we can use d to adjust the value of α at Q', it suffices to show that we can choose u, v such that $[(x-u)/(x-v)]/[(y-u)/(y-v)] = c$. But the left-hand side equals $[(x-u)/(y-u)][(y-v)/(x-v)]$ and $(x-u)/(y-u) = 1 + (x-y)/(y-u)$ which as we vary u is an arbitrary k'' element not including 1. Arguing similarly for the denominator, we are done. □

We are ready for the promised result

Proposition 3.8.

(a) *Suppose $P = \mathrm{Spec}(k'')$ is a closed point in the exceptional divisor of $X \to \mathrm{Spec}(R)$. Then there is a unipoint prime $\delta \in R$ such that if D is the strict transform of δ, then D is nonsingular at P and has arbitrary k'' rational slope.*
(b) *Suppose $E \subset X$ is an exceptional line and a limb. Let P as above be a point on E and not on any other exceptional line. Assume k'' has more than three elements. Let $c \in k''^*$. Then there is a $\beta \in q(R)$ such that the following holds. First, β is defined and has value c at P, and is defined and has constant value 1 on every other exceptional line in X. Second, the support of (β) consists of unipoint prime ideals whose strict transforms are nonsingular in X and intersect only E.*

Proof. If we blow-up the point P then part (a) is obvious from 3.4. As for (b), let Q be the closed point on E which intersects the rest of the exceptional lines. Choose α as in 3.7 with value 1 at Q and value c at P. Let P_1, Q_1 be the zero and pole of α on E. Let C_1 and D_1 be ordinary unipoint curves which contain P_1 and Q_1 respectively, are nonsingular there, and have tangents distinct from E. By 2.3 there is a $\beta \in q(R)$ with divisor $C_1 - D_1$ on X. It follows that β has no zeroes and poles on any other exceptional line of X other than E, and so β is constant on those lines. Since one of those lines is defined over k, it follows that this constant is in k. By adjusting β by a unit of R, we can assume β has constant value 1 on all the other exceptional lines of X. The restriction of β to E has the same zero and pole as α, and the same value at Q, and so must equal α. □

4. General results

In this section we prove some general results about ramification as well as results about lifting cyclic extensions and Brauer group elements. We begin with the ramification result. For any ring S, let $J(S)$ be its Jacobson radical.

Theorem 4.1. *Suppose R is an excellent local integrally closed two dimensional domain with $K = q(R)$. Let R'/R be finite etale. Let π' be a prime ideal of R', $\tilde{R}' = R'/\pi'$, and $\bar{R}'_\pi$ the integral closure of $\tilde{R}'$ in $q(\tilde{R}')$. Let $\bar{T}/\bar{R}'_{\pi'}$ be a cyclic Galois extension of prime power degree n. Then there is an $\alpha \in \mathrm{Br}(q(R'))$ which ramifies at no prime of R' except π' and at π' has ramification $q(\bar{T})/q(\tilde{R}'), \sigma$ for some generator σ.*

Proof. Let k be the residue field of R. There are technical difficulties when k is a finite field, so to address these first we assume k finite. Let k''/k be a finite extension of fields of degree prime to n such that $k'' = k[t]/h(t)$ and let $f(t)$ be a monic lift of $h(t)$. Set $R'' = R[t]/(f(t))$. Note that

$f(t)$ is separable so R''/R is etale. Then $R' \otimes_R R''$ is a direct sum of domains etale finite over R, and so we can choose one, R_3, such that $q(R_3)/q(R')$ has degree prime to n. If π_3 is a prime of R_3 over π', let $\bar{R}_3$ be the integral closure of R_3/π_3 in $q(R_3/\pi_3)$. If $\tilde{f}(t) \in \tilde{R}$ is the image of $f(t)$, then $\tilde{f}(t)$ is separable. It follows that $\bar{R}_3$ is an image of $\bar{R}'_\pi[t]/(\tilde{f}(t))$. If $\bar{R}'_{\pi'}/J(\bar{R}'_{\pi'}) \cong \bigoplus_{i=1}^r k_i$, then $\bar{R}_3[t]/J(\bar{R}_3[t])$ is an image of $\bigoplus k_i[t]/h(t)$. Note that $k_i[t]/(h(t))$ is a direct sum of fields and the number of fields in this direct sum is bounded by the degree of k_i/k. Furthermore, using corestriction, to prove the result for R' it suffices to prove it for R_3 viewed as an extension of R''. Thus, by making the degree of k''/k large enough, we may assume:

(I) The cardinality of k is greater than nr, where r is the number of maximal ideals of $\bar{R}'_{\pi'}$.

Now we return to the case of general k and prove the theorem in the following special case:

Lemma 4.2. *Theorem* 4.1 *is true when $\bar{T}$ has the form $T \otimes_{\tilde{R}'} \bar{R}'_\pi$ for $T/\tilde{R}'$ cyclic Galois of degree n.*

Proof. Let J be the Jacobson radical of $\tilde{R}'$ and $T_J = T \otimes_{\tilde{R}'} \tilde{R}'/J$. By (I), we have that $T_J = (\tilde{R}'/J[t])/(\bar{g}(t))$ for a monic separable polynomial $\bar{g}(t)$ of degree, say, s. Let $\bar{\alpha} \in T_J$ be the image of t and $\alpha \in T$ a preimage of $\bar{\alpha}$. By Nakayama's Lemma, $T = \tilde{R}' + \tilde{R}'\alpha + \cdots + \tilde{R}'\alpha^{s-1}$, so $T = \tilde{R}'[t]/(\tilde{g}(t))$ where $\tilde{g}(t)$ is a monic preimage of $\bar{g}(t)$.

Let $M_1, \ldots, M_s$ be the maximal ideals of R' not containing π' and $I = M_1 \cap \cdots \cap M_s$. Then $(\pi') + I = R'$. It follows that there is a lift $g(t) \in R'[t]$ which is a preimage of $\tilde{g}(t)$ and maps to a separable polynomial in R'/I. Thus the discriminant of $g(t)$ is a unit and $S = R'[t]/(g(t))$ is etale over R'.

Let $\bar{L}/q(R')$ be the Galois closure of $q(S)/q(R')$ with Galois group G and $\bar{S}$ the integral closure of R' in $\bar{L}$. Note that $\bar{S}/R'$ is etale and G Galois because $\bar{L}$ is formed by adjoining roots of factors of $g(t)$, which must have unit discriminants. Alternatively, one can form the Galois closure $\bar{S}'$ as in [LN, p. 42], which has S_n Galois group, and note that being etale over R', $\bar{S}'$ must be the direct sum of domains, any of which we can take for $\bar{S}$ with Galois group $G \subset S_n$.

Let $\bar{\pi}$ be a prime of $\bar{S}$ over π'. Then the stabilizer of $\bar{\pi}$ is $C_n \subset G$, the cyclic group of order n and can be identified with the Galois group of $q(T)/q(\tilde{R}')$. Let $S'' = \bar{S}^{C_n}$, $L'' = q(S'')$, and let $\pi'' \in S''$ be the prime defined by $\bar{\pi}$. Then $q(S''/\pi'') = q(\tilde{R}')$ and $q(\bar{S}/\bar{\pi}) = q(T)$. Form the cyclic algebra $\Delta(\bar{L}/L'', \sigma, \pi'')$ with Brauer class β. Since this algebra is Azumaya at all primes except π'', β is only ramified at π'' with ramification $q(T)/q(\tilde{R}')$. Let α be the corestriction of β to $q(R')$. By e.g. 0.1, α is the Brauer class we need. $\square$

For the rest of the proof of 4.1, we try and reduce the general case to that of 4.2. To this end, we need to understand when an extension $\bar{T}/\bar{R}'_{\pi'}$ comes from $\tilde{R}' = R'/\pi'$, meaning $\bar{T} = T \otimes_{\tilde{R}'} \bar{R}'_{\pi'}$ for some etale $T/\tilde{R}'$. Let I be the conductor of $\tilde{R}' \subset \bar{R}'_{\pi'}$. We have the pullback square:

$$\begin{array}{ccc} \tilde{R}' & \subset & \bar{R}'_{\pi'} \\ \downarrow & & \downarrow \\ \tilde{R}'/I & \subset & \bar{R}'_{\pi'}/I. \end{array}$$

Let $\tilde{J} \subset \tilde{R}'$ and $\bar{J} \subset \bar{R}'_{\pi'}$ be the respective Jacobson radicals and set $\bar{T}_I = \bar{T}/I\bar{T}$. We have the well known:

Theorem 4.3.

(a) *$\bar{T}$ comes from $\tilde{R}'$ if and only if $\bar{T}/\bar{J}\bar{T}$ comes from $\tilde{R}'/\tilde{J}$.*
(b) *If $\bar{T}/\bar{R}'_{\pi'}$ is a G Galois extension, and $\bar{T}/\bar{J}\bar{T}$ comes from a G Galois extension of $\tilde{R}'/\tilde{J}$, then $\bar{T}$ comes from a G Galois extension of $\tilde{R}'$.*

Proof. Of course the condition in (a) is necessary, so assume $\bar{T}/\bar{J}\bar{T}$ comes from $\tilde{R}'/\tilde{J}$ and prove (a) and (b) simultaneously. Then $\tilde{J}/(I \cap \tilde{J})$ and $\bar{J}/(I \cap \bar{J})$ are both nilpotent, so $\bar{T}/(I \cap \bar{J})(\bar{T})$ comes from $\tilde{R}'/(I \cap \tilde{J})$. Going modulo I, we have that there is a finite etale extension T_I of $\tilde{R}'/I$ and an $\bar{R}'_{\pi'}$ isomorphism $\phi : T_I \otimes_{\tilde{R}'/I} \bar{R}'_{\pi'}/I \cong \bar{T}_I$. It is clear that in case (b) we can assume $T_I/(\tilde{R}'/I)$ is a G Galois extension and ϕ preserves the G action.

Set $T \subset (T_I \times \bar{T})$ to be the subring of pairs $(a_I, \bar{a})$ such that $\phi(a_I \otimes 1) = \bar{a} + I\bar{T}$. In (b), T has inherited a G action. Since $\bar{T}$ and T_I are free modules of the same rank, it is clear that T is a free module over $\tilde{R}$. Thus using ϕ there is a pullback square:

$$\begin{array}{ccc} T \otimes_{\tilde{R}'} T & \subset & \bar{T} \otimes_{\bar{R}'_{\pi'}} \bar{T} \\ \downarrow & & \downarrow \\ T_I \otimes_{\tilde{R}'/I} T_I & \subset & \bar{T}_I \otimes_{\bar{R}'_{\pi'}/I} \bar{T}_I. \end{array}$$

Since the separating idempotents $e \in \bar{T} \otimes_{\bar{R}'_{\pi'}} \bar{T}$ and $e_I \in T_I \otimes_{\tilde{R}/I} T_I$ are unique, they define one in $T \otimes_{\tilde{R}} T$ and $T/\tilde{R}'$ is etale. In a similar way, in (b), $T/\tilde{R}'$ is G Galois. □

The strategy is now to enlarge $\tilde{R}'/\tilde{J}$ in order to apply 4.3 and 4.2. We use a trick we will have future need for, so we separate out the argument as a distinct result.

Proposition 4.4. *Let R', $\tilde{R}' = R'/\pi'$, and $\tilde{R}/\tilde{J} \subset \bar{R}'_{\pi'}/\bar{J} = \bigoplus_i k_i$ be as above. Suppose $k \subset k'_i \subset k_i$ are such that k'_i/k are separable and that, as a subring of $\bar{R}'_\pi/\bar{J} = \bigoplus_i k_i$, $\bigoplus_i k'_i = k[\bar{\theta}]$. Let $\bar{g}(t)$ be the minimum polynomial of $\bar{\theta}$ over $\tilde{R}/\tilde{J}$.*

(a) *Let $\theta \in \bar{R}'_\pi$ be a preimage of $\bar{\theta}$. Then there is monic preimage $\tilde{g}(t) \in \tilde{R}'[t]$ such that $\tilde{g}(\theta) = 0$.*
(b) *If $g(t) \in R[t]$ is a monic lift of $\tilde{g}(t)$, $S = R[t]/(g(t))$ is etale over R. Furthermore, $S \otimes_R R'$ has a direct summand of the form $R'' = R'[t]/(g_1(t))$ such that R'' is regular, etale over R', and the following holds. R'' has a prime π'' over π' such that π'' has the following properties*:
   (i) *$\tilde{R}' \subset R''/\pi'' \subset \bar{R}'_{\pi'}$ and $\theta \in R''/\pi''$.*
   (ii) *$k \subset \tilde{R}'/\tilde{J} \subset \bigoplus_i k'_i \subset R''/J'' \subset \bar{R}'_{\pi'}/\bar{J} = \bigoplus_i k_i$.*
   Furthermore, we have:
(c) *Let $L/q(R'/\pi'), \sigma$ be a cyclic Galois extension, and $\delta'_1, \ldots, \delta'_r \in R'$ be a set of primes (perhaps empty) which extend to primes $\delta''_1, \ldots, \delta''_r \in R''$. Let $g_1(t)$, π'' and R'' be as in* (a) *and* (b). *Suppose there is an $\alpha'' \in \mathrm{Br}(q(R''))$ which ramifies at π'' with ramification $L/q(R''/\pi''), \sigma$ and whose other ramification is only among the δ''_j. Then there is an $\alpha' \in \mathrm{Br}(q(R'))$ whose ramification at π' is $L/q(R''/\pi''), \sigma$ and all of whose other ramification is at the δ'_j.*

Proof. Let $\pi \in R$ be the prime under π'. By Nakayama'a Lemma, $(R/\pi)[\theta]$ is spanned over R/π by $1, \theta, \ldots, \theta^{s-1}$ where s is the degree of $\bar{g}(t)$. Then the existence of $\tilde{g}(t)$ is clear, proving (a).

As for (b), since $\bar{g}(t)$ is separable, S/R is etale and $S \otimes_R R' = \bigoplus_j R'[t]/(g_j(t))$ where the $g_j(t)$ are irreducible and $\prod_j g_j(t) = g(t)$. It follows that $\tilde{g}(t) = \prod_j \tilde{g}_j(t)$ and θ is a root of one of them, say $\tilde{g}_1(t)$. We let $R'' = R''_1$ and let π'' be the prime of R'' corresponding to $t - \theta$ as a root of $\tilde{g}_1(t)$. The rest of (b) is now clear. Let α'' be as in (c) and set $\alpha' = \mathrm{Cor}_{q(R'')/q(R')}(\alpha'') \in \mathrm{Br}(q(R'))$. Then by 0.1, (c) follows. $\square$

Returning to the proof of 4.1, $\bar{R}'_{\pi'}/\bar{J} = \bigoplus_i k_i$ is a direct sum of fields and $\bar{T}/\bar{J}\bar{T}$ is a direct sum of cyclic Galois extensions of these k_i. It follows that there are subfields $k'_i \subset k_i$ such that the k'_i are separable over k and $\bar{T}/\bar{J}\bar{T}$ comes from $\bigoplus_i k'_i$. Using (I) there is a $\bar{\theta} \in \bigoplus_i k'_i$ such that $\bigoplus k'_i = k[\bar{\theta}]$. We can apply 4.3 and 4.2 to the R'' in 4.4, and 4.1 is proven.

In the next section we prove a result about the surjectivity of $\mathrm{Br}(q(R)) \to \mathrm{Br}(q(R^h))$, where R^h is the henselization. It turns out that a key aspect of this result is the need to lift cyclic extensions and Brauer group elements over local rings. The lifting results we need are known in the characteristic equal case, but not in general—largely as an oversight. The next theorem covers that case also.

Theorem 4.5. *Suppose R is a semilocal integrally closed domain with Jacobson radical J and $R/J = l = \bigoplus_i k_i$ for fields k_i. Let q be a prime power which is prime to the characteristic of all k_i. If q is odd, then*:

(a) *If $\alpha \in \mathrm{Br}(l)$ has order dividing q, then α has a preimage in* $\mathrm{Br}(R)$ *of order dividing q.*
(b) *If L/l is a cyclic Galois extension of degree q, there is a cyclic Galois T/R such that $T \otimes_R l \cong L$, the isomorphism preserving the Galois group actions.*

If $q = 2^n$, and $k_i(\rho)/k_i$ is cyclic for all i where ρ is a primitive q root of one, then (a) *and* (b) *still hold.*

Proof. For all prime power q, if ρ is a primitive q root of one, then $k_i(\rho)/k_i$ is cyclic. To prove 4.5 we begin with a special case:

Lemma 4.6. *If $\rho \in R$,* 4.5 *holds.*

Proof. In (a), α is a product of symbol algebras $(\bar{a}, \bar{b})_{n,k}$ and if we choose $a, b \in R$ preimages of $\bar{a}, \bar{b}$, then the Azumaya symbol algebra $(a, b)_{n,R}$ is a lift for $(\bar{a}, \bar{b})_{n,k}$ and (a) is clear. Part (b) is the same using Kummer theory. $\square$

We return to prove the general case of 4.5. Let $f(t)$ be the minimum monic polynomial of ρ over $q(R)$, which must have coefficients in R since R is integrally closed. Form $R' = R[t]/(f(t))$. Then R' is a semilocal domain which is again integrally closed since it is etale over R. If J' is the Jacobson radical of R', then $R'/J' = \bigoplus_i k_i[t]/(f_i(t))$ where $f_i(t)$ is the image of $f(t)$ in $k_i[t]$. Thus R'/J' is a direct sum of fields k_{ij} where $k_{ij} = k_i(\rho)$ is a cyclic extension of k_i. The key observations are:

Proposition 4.7. *Suppose F is a field of characteristic prime to q, ρ is as above, and $F(\rho)/F$ is a cyclic Galois extension. Suppose $\tau(\rho)=\rho^r$ generates the Galois group of $F(\rho)/F$.*

(a) *If q is odd, or r is not congruent to -1 modulo q, then every element of* $\mathrm{Br}(F)$ *of order dividing q is the corestriction of an element of* $\mathrm{Br}(F(\rho))$. *In the same way, every cyclic Galois $L/F,\sigma$ of degree q is the corestriction of a cyclic Galois $L'/F(\rho),\sigma'$ also of degree q.*
(b) *If q is even and r is congruent to -1 modulo q, then every element of* $\mathrm{Br}(F)$ *is a corestriction from* $\mathrm{Br}(F(\rho))$ *times a product of quaternion algebras. Every degree q cyclic Galois extension $L/F,\sigma$ is a product of a corestriction of a degree q cyclic over $F(\rho)$ and a quadratic extension.*

Proof. The Brauer group cases of (a) and (b) are in [Me]. The corresponding result [S0] for cyclic Galois extensions is in a different language, so we need to make a few comments.

Set $F'=F(\rho)$. Let τ have order m, so r^m-1 is divisible by q. In (a), we can assume (e.g. [S0, p. 257]) that $(r^m-1)/q$ is prime to q. In (b), we can assume $r=-1$.

If $L'/F(\rho),\sigma'$ is a cyclic Galois extension of degree q, then $L'=F'(a^{1/q})$ where $\sigma'(a^{1/q})=\rho a^{1/q}$. Thus writing L' as $F(a^{1/q})$ specifies both the extension and the generator of the Galois group. The τ conjugate of L'/F' then has the form $F'(\tau(a)^{s/q})$, where $rs-1$ is divisible by q. We begin with (a). Up to q powers, we can write $a=b^{r^{m-1}}$ and so $\tau(a)^s$ as $\tau(b)^{r^{m-2}}$. It follows that the corestriction, $L/F,\sigma$, of $L'/F',\sigma'$ can be described as follows. Consider $L''=F'(M^{1/q})$ where $M=b^{r^{m-1}}\tau(b)^{r^{m-2}},\dots,\tau^{m-1}(b)$. Then τ extends to L'' by setting $\tau(M^{1/q})$ to have the form $M^{r/q}/b^{(r^m-1)/q}$. There is a choice here so that the extension of τ has order m and L is the τ fixed field of L''. But this is just the description of all cyclic Galois extensions in [S0, p. 258]. and so (a) is done.

In (b), the τ conjugate of L'/F' is $F'(\tau(a)^{-1/q})$ and so the corestriction is defined as follows. Let $L''=F'(M^{1/q})$ where $M=a/\tau(a)$ and extend τ by setting $\tau(M^{1/q})=1/M^{1/q}$. The corestriction L/F is the τ fixed field. In [S0, p. 258] it is shown that all L/F have the following form. Let $b\in F$ and $a\in F'$, and form $L''=F'(M^{1/q})$ where $M=b^{n/2}\tau(a)/a$. Then τ extends to L'' by setting $\tau(M^{1/q})=(M^{1/q})^{-1}b$, and L is the τ fixed field. Then (b) follows. $\square$

We can now prove 4.5 in general. We will do the cyclic Galois case, as the Brauer group case is exactly the same. Let $J'\subset R'$ be the Jacobson radical and $l'=R'/J'=\bigoplus_{ij}k_{ij}$. If L/l is cyclic Galois of degree n, then $L=\bigoplus L_i$ where L_i/k_i is cyclic Galois of degree n. Since quadratic extensions lift (4.6), we may assume each L_i is the corestriction of an L_{i1}/k_{i1}. Let $L'_i=\bigoplus_j L_{ij}$ where L_{i1}/k_{i1} is as given and L_{ij}/k_{ij} is split for $j>1$. Set $L'=\bigoplus_i L'_i$, a cyclic degree n Galois extension of l'. Then the corestriction of L'/l' is L/l. By 4.6 L'/l' lifts to a cyclic Galois T'/R' and taking corestriction we are done. $\square$

5. A surjectivity result

This section is the core of the paper. As always, let R be a regular local ring of dimension two. In 1.1 we proved that the composition $\mathrm{Br}(q(R))\to\bigoplus_{P\subset R}H^1(q(R/P),\mathbb{Q}/\mathbb{Z})\to\mu^{-1}$ is the zero map. In this section we show this is exact. A key part of the argument involves the relationship between $\mathrm{Br}(q(R))$ and $\mathrm{Br}(q(R^h))$, and in the process we noticed the surprising fact that $\mathrm{Br}(q(R))\to\mathrm{Br}(q(R^h))$ is surjective most of the time. Of the two results, we feel the second is more fundamental. These two results are the focus of this section, and intersection of the two proofs is quite large.

Theorem 5.1. *Let R be a regular excellent 2 dimensional local ring with field of fractions $q(R)$. Let R^h be the henselization and p the characteristic (perhaps 0) of the residue field of R. When the characteristic of $q(R)$ is not 2, assume that, for all n, adjoining a 2^n root of one to $q(R)$ is a cyclic extension of $q(R)$. Then the natural restriction map $\mathrm{Br}(q(R)) \to \mathrm{Br}(q(R^h))$ of Brauer groups is surjective on elements of prime to p order.*

Remark. Looking at the proof to come, we note that the hypothesis of 5.1 about 2^n roots of one can be dropped if we restrict ourselves to odd order elements.

Theorem 5.2. *Let R be a regular excellent 2 dimensional local ring with field of fractions $q(R)$. Then the sequence*

$$\mathrm{Br}\big(q(R)\big)' \to \bigoplus_{P\subset R} H^1\big(q(R/P), \mathbb{Q}/\mathbb{Z}\big)' \to \mu^{-1}$$

is exact.

The proof of these results will take most of this section. In both cases, the main work will be in analyzing which elements of $\bigoplus_{P\subset R^h} H^1(q(R^h/P), \mathbb{Q}/\mathbb{Z})$ come from $\mathrm{Br}(q(R^h))$ and $\mathrm{Br}(q(R))$. Often, we will be able to reduce to elements of $\bigoplus_{P\subset R^h} H^1(q(R^h/P), \mathbb{Q}/\mathbb{Z})$ which either are nonzero at one prime, or at one prime and at one fixed nonsingular prime.

We begin with a definition. Let R be as above, and π a prime of R. We say $L/q(R/\pi), \sigma$ is unramified if this extension is unramified with respect to the integral closure $\bar{R}_\pi$. Note that this means $L = \bar{T} \otimes_{\bar{R}_\pi} q(\bar{R}_\pi)$ where $\bar{T}/\bar{R}_\pi, \sigma$ is cyclic Galois. Note also that this is stronger than saying the ramification map r_π sends this extension to 1, both because $\bar{R}_\pi$ can have multiple primes whose effects cancel, and because the residue fields of $\bar{R}_\pi$ are larger than k. In 5.5 we note that for so-called purely inseparable primes of R^h (including nonsingular primes), mapping to $1 \in \mu^{-1}$ is equivalent to being unramified.

If δ is a nonsingular prime of R, then δ is also a nonsingular prime of R^h and they all arise in this way (up to units). Some of the detailed results we need are:

Proposition 5.3.

(a) *Suppose $\pi^h \in R^h$ is a prime and $\delta \in R$ is a prime nonsingular at the closed point. Let $L/q(\bar{R}^h_{\pi^h}), \sigma$ be a cyclic extension of degree q^r where q is a prime unequal to the characteristic of k. Then there is an $\alpha^h \in \mathrm{Br}(q(R^h))$ such that α^h only ramifies at π^h and δ, and the ramification of α^h at π^h is $L/q(R^h/\pi^h), \sigma$.*

(b) *Suppose π^h lies over the prime π of R. In* (a), *there is an $\alpha \in \mathrm{Br}(q(R))$ with image $\alpha'^h \in \mathrm{Br}(q(R^h))$ such that α only ramifies at π and δ and the ramification of $\alpha^h - \alpha'^h$ is itself unramified.*

To prove 5.3, let $k = R/M = R^h/M^h$ be the residue field of R and R^h. As usual for any prime π, let $\bar{R}_\pi$ be the integral closure of R/π in $q(R/\pi)$. Let π^h be a prime of R^h. Let $\bar{R}^h_{\pi^h}$ be the integral closure of $R^h/(\pi^h)$. By 0.4 $\bar{R}^h_{\pi^h}$ is a local ring with residue field we write as k_π. Since in (a) we may take $R = R^h$, we can assume π^h lies over a prime π of R. By 1.3 and direct limits, $\bar{R}_\pi \subset \bar{R}^h_{\pi^h}$ and the maximal ideal of $\bar{R}^h_{\pi^h}$ lies over a maximal ideal P of $\bar{R}_\pi$ with the same residue field k_π. It follows that $\bar{R}^h_{\pi^h}$ is the henselization of $\bar{R}_\pi$ localized at P.

First we observe we can enlarge k if needed. Suppose $k' = k(\bar{\theta})$ is a finite separable extension of k of degree prime to q and $\bar{g}(t)$ is the minimal monic polynomial of $\bar{\theta}$ over k. Let $g(t) \in R[t]$ be a monic preimage and $R' = R[t]/(g(t))$. Of course, R' is a regular local 2 dimensional excellent domain. Choose π' a prime of R' over π such that $[q(R'/\pi') : q(R/\pi)]$ is prime to q.

Lemma 5.4. *Proving* 5.3 *for R' and π' implies it for R and π.*

Proof. Clearly $R'^h = R^h[t]/(g(t))$ and π' extends to a prime π'^h of R'^h such that π'^h lies over π^h and $[q(R'^h/\pi'^h) : q(R^h/\pi^h)]$ is prime to q. Using corestriction parts (a) and (b) are clear. $\square$

Because of 5.4 we can assume k has cardinality larger than any fixed number and we do so without further comment. Returning to the main line of our argument, of course k_π/k is a finite extension. If k_π/k is purely inseparable, we say π^h is purely inseparable. If π is unipoint and π^h is purely inseparable, we say π is purely inseparable. Let us note some easy properties of purely inseparable unipoint primes.

Lemma 5.5. *Suppose π is a unipoint purely inseparable prime with extension π^h.*

(a) *Let ρ be a root of unity of order prime to p. Then $\rho \in k_\pi$ implies $\rho \in k$ which implies $\rho \in R^h$.*
(b) *Let $L/q(R/\pi), \sigma$ have degree q prime to p and ramification $\rho \in \mu^{-1}$. Then the order of ρ is the ramification index of $L/q(R/\pi)$ with respect to the discrete valuation domain $\bar{R}_\pi$.*
(c) *In* (b)*, if $\rho = 1$ then $L/q(R/\pi)$ is unramified and comes from a cyclic extension of R/π.*

Proof. Part (a) is obvious, and (b) is not much harder since k_π/k has degree prime to q. As for (c), we know $L/q(R/\pi)$ comes from a cyclic extension $\bar{T}/\bar{R}_\pi$. Since every degree q cyclic extension of k_π comes from k, we are done by 4.3. $\square$

Using corestriction we can reduce 5.3 to the case π^h is purely inseparable.

Proposition 5.6. *In order to prove* 5.3*, it suffices to consider the case π^h is purely inseparable.*

Proof. Let $\bar{\theta} \in k_\pi$ be such that $k(\bar{\theta})$ is the maximal separable extension in k_π/k. If $\bar{g}(t)$ is the minimum polynomial of $\bar{\theta}$ let $g(t) \in R[t]$ be a monic preimage. Set $R' = R[t]/(g(t))$ which has henselization $R'^h = R^h[t]/(g(t))$ since $g(t)$ is necessarily irreducible over R^h. But $\bar{R}^h_{\pi^h}$ is henselian so if $\tilde{g}(t) \in (R/\pi)[t]$ is the image of $g(t)$, there is a $\tilde{\theta} \in \bar{R}^h_{\pi^h}$ which is a root of $\tilde{g}(t)$ and maps to $\bar{\theta}$. Let $\pi'^h \in R'^h$ be the prime associated to the factor $t - \tilde{\theta}$ of $\tilde{g}(t)$ and π' the restriction of π'^h to R'. Then $q(R'^h/\pi'^h) = q(R^h/\pi^h)$ so $\bar{R}^h_{\pi^h}$ is the integral closure of R'^h/π'^h. In particular, π'^h is purely inseparable. The nonsingular prime δ extends uniquely to a primes of R^h, R', and R'^h and we call all these primes δ. By assumption, there is an $\alpha^{h'} \in \mathrm{Br}(q(R'^h))$ with ramification only at π'^h and δ and ramification $L/q(\bar{R}^h_{\pi^h}), \sigma$ at π'^h. In particular, $\alpha^{h'}$ does not ramify at any other extensions of π^h. Taking corestrictions and using 0.1, we have shown (a). In (b), suppose $\alpha' \in \mathrm{Br}(q(R'))$ is such that its image and $\alpha^{h'}$ only differ by unramified ramification. Since this property is preserved by corestriction, (b) is clear. $\square$

Of course, because of 5.6, we now assume π^h is purely inseparable.

Lemma 5.7. *Assume $\pi^h \in R^h$ is a purely inseparable prime and $L^h/q(\bar{R}^h_{\pi^h}), \sigma$ is a cyclic Galois extension of degree, n, prime to p. Let ρ be the ramification of this cyclic extension as defined in Section 1. Suppose π is a prime of R lying under π^h. Set $P \subset \bar{R}_\pi$ to be the maximal ideal induced by $\bar{R}^h_{\pi^h}$ and R_P the localization of $\bar{R}_\pi$ at P.*

(a) *There is a cyclic extension $L_2/q(\bar{R}_\pi), \sigma_2$ with ramification ρ at R_P and which is unramified at all the other maximal ideals of $\bar{R}_\pi$.*
(b) *Let $L^h_2/q(\bar{R}^h_{\pi^h})$, σ_2 be the element of $H^1(q(\bar{R}^h_{\pi^h}), \mathbb{Q}/\mathbb{Z})$ which is the image of the extension in* (a). *Then $L^h_2/q(\bar{R}^h_{\pi^h})$, σ_2 has ramification ρ at the closed point. Furthermore, $L^h/q(\bar{R}^h_{\pi^h}), \sigma$ is the product of a cyclic extension $L^h_1/q(\bar{R}^h_{\pi^h}), \sigma_1$ and $L^h_2/q(\bar{R}^h_{\pi^h})$, σ_2 where $L^h_1/q(\bar{R}^h_{\pi^h})$ is unramified.*

Proof. Let n_2 be the order of ρ which by 5.5 is the ramification index of $L^h/q(\bar{R}_{\pi^h})$. It follows that $\rho \in k$. Then part (a) follows from the following result 5.8. To state this result, suppose T is a semilocal Dedekind domain with field of fractions K and let P be a maximal ideal of T. Suppose ρ is a root of unity of order n in T/P (implying that n is prime to the characteristic of this field and K). Assume all the residue fields of T have the same characteristic. Let $T' = T[\rho]$ and let $P' \subset T'$ be a prime ideal over P. Since T'/T is etale, T' is a semilocal Dedekind domain and hence is a UFD. In particular, there is a $\delta' \in T'$ which has valuation one with respect to P' and is a unit at all the other prime ideals of T'.

Lemma 5.8. *For any such δ', let $L' = q(T')(\delta'^{1/n})$ with the canonical generator σ' of its Galois group over $q(T')$. Let $L/q(T), \sigma$ be the corestriction of $L'/q(T'), \sigma'$. Then this extension has ramification ρ at P and is unramified at all the other prime ideals of T.*

Proof. We may assume n is a prime power. Let $K' = K(\rho)$. T' is the integral closure of T in K'. The prime ideal P of T splits completely in T'. If G is the Galois extension of K'/K, then G is generated either by τ such that $\tau(\rho) = \rho^m$ and $\rho^m \neq \rho^{-1}$, or by σ such that $\sigma(\rho) = \rho^{-1}$, or by both such a τ and such a σ. Call these cases 1, 2, or 3. In cases 1 and 3 we can let s be the order of m modulo q and choose the integer m such that $(m^s - 1)/q$ is prime to q (e.g. [S0, p. 257]). In these cases define $M_\tau(x) = x^{m^{s-1}} \tau(x)^{m^{s-2}}, \ldots, \tau^{s-1}(x)$. Set $b = M_\tau(\delta')$, or $b = \sigma(\delta')/\delta'$, or $b = M_\tau(\sigma(\delta')/\delta')$ in cases 1, 2, 3, respectively. Let $L' = K'(b^{1/n})$. By [S0, pp. 258–260], in all cases $L' = L \otimes_K K'$ where L/K is cyclic Galois. It is clear that L/K is ramified as described. By the argument in 4.7 L/K is the corestriction of L'/K' as required. $\square$

We return to the proof of 5.7, specifically part (b). Let δ be a prime element of $\bar{R}_\pi$ defined by the maximal ideal of $\bar{R}^h_{\pi^h}$. Then δ is prime in $\bar{R}^h_{\pi^h}$ and the first statement of (b) is clear. $L^h_1/q(\bar{R}^h_{\pi^h}), \sigma_1$ can be defined as the product of $L/q(\bar{R}^h_{\pi^h}), \sigma$ and the inverse of $L^h_2/q(\bar{R}^h_{\pi^h}), \sigma_2$. It is unramified by 5.5. $\square$

Thus to prove 5.3(a) in the case π^h is purely inseparable we can consider two distinct cases. One case will be when $L/q(\bar{R}^h_{\pi^h})$ is unramified and the other is 5.10 to follow. Let us settle the easy unramified case first, but state the result more generally.

Lemma 5.9. *Suppose π is unipoint purely inseparable and $L/q(R/\pi), \sigma$ maps to $1 \in \mu^{-1}$. Then there is an $\alpha \in \mathrm{Br}(q(R))$ which ramifies only at π with ramification $L/q(R/\pi), \sigma$.*

Proof. By 5.5 $L/q(R/\pi)$ is unramified at $\bar{R}_\pi$ and we are done by 4.1. □

Since all primes of R^h are unipoint, 5.9 plus 5.10 finishes 5.3(a). In addition, 5.3(b) is also immediate from 5.10, to which we turn. Let $\pi \in R$ be a prime which lies under the prime π^h of R^h. Let δ be a nonsingular prime of R. Set $\bar{R}_\pi$ to be the integral closure of R/π and $P \subset \bar{R}_\pi$ the maximal ideal associated to π^h as in 1.4.

Proposition 5.10. *Let n be a prime power not divisible by the characteristic of k and ρ a primitive n root of one in $\bar{R}_\pi/P$. Let π, π^h, etc. be as above. In particular, we assume π^h is purely inseparable.*

(a) *There is an $\alpha \in \mathrm{Br}(q(R))$ with the following properties. α only ramifies at π and δ. The ramification of α at π is itself unramified at all the prime ideals of $\bar{R}_\pi$ except P, and at P this ramification itself has ramification ρ.*
(b) *Let $\alpha^h \in \mathrm{Br}(q(R^h))$ be the image of α. Then α^h has the following properties. α^h is only ramified at the extensions of π and at δ. At all the extensions of π except π^h the ramification of α^h is itself unramified, and at π^h this ramification itself has ramification ρ.*

Proof. Part (b) follows from (a). Also, since π^h is purely inseparable, $\rho \in k$ so $\rho \in R^h$. It will take some time to prove 5.10(a). At this point it is convenient to observe:

Lemma 5.11. *Suppose* 5.10(a) *holds for a unipoint purely inseparable prime π. Let $L/q(R/\pi)$, σ be any cyclic extension of degree n prime to p. Then there is an $\alpha \in \mathrm{Br}(q(R))$ which ramifies only at π and δ and whose ramification at π is $L/q(R/\pi), \sigma$.*

Proof. Suppose $\alpha' \in \mathrm{Br}(q(R))$ ramifies only at π and δ and its ramification at π itself has ramification of order n at $\bar{R}_\pi$. Then after modifying by a power of α', we can assume $L/q(R/\pi)$, σ is unramified and we are done by 5.9. □

Before we can say more we need to consider what happens when we add a root of one, ρ, to R. For the moment we do not assume $\rho \in k$.

Let ρ be a primitive n root of one over R, where n is a prime power not divisible by the characteristic of k. Suppose $f(t) \in R[t]$ is the minimum monic polynomial of ρ over R. Set $R' = R[t]/(f(t))$ which is Galois over R with abelian group H. We also write $R' = R(\rho)$. If $\bar{H}$ is the Galois group of $k(\rho)/k$, then we can identify $\bar{H}$ with a subgroup of H. R' is semilocal and modulo its Jacobson radical it is the direct sum of $[H : \bar{H}]$ copies of $k(\rho)$. We set $R^* = R' \otimes_R R^h$ and note by the henselian property that R^* can be written as a direct sum of $[H : \bar{H}]$ copies of $R'^h = R^h(\rho)$.

If π is a prime of R let $H_\pi \subset H$ be the stabilizer of some and hence all extensions of π to R'. If π extends to a purely inseparable prime π^h, it is easy to see that $\bar{H} \subseteq H_\pi \subseteq H$, but otherwise there is no obvious connection between $\bar{H}$ and H_π. Let $\pi' \in R'$ generate a prime over π. Consider a sequence of blow-ups $X \to \mathrm{Spec}(R)$ such that the strict transform, C, of $\pi = 0$ is nonsingular, meets each exceptional line of $X \to \mathrm{Spec}(R)$ in at most one point, and each point of C is on only one exceptional line. Form $X' = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$. Set $\bar{X} \subset X$ to be the closed fiber. Let $M_1, \dots, M_s \subset R'$ be the maximal ideals of R' which, of course, lie over the maximal ideal M of R. Then $\bar{H}$ is the stabilizer of any of the M_i. Choose coset representatives of $\bar{H}$ in H, $1 = h_1, h_2, \dots, h_s$, numbered so that $M_i = h_i(M_1)$. Choose $f \in M_2 \cap \cdots \cap M_s$ with $f \notin M_1$. Set $U_1 = \operatorname{Spec}(R'(1/f))$ and $U_i = h_i(U_1)$. Then it is clear that the U_i form an affine open cover of $\operatorname{Spec}(R')$ and that U_i contains M_i and none of the other M_j's. It is also clear that X' can be obtained by patching sequences of point blow-ups $X_i \to U_i$ where X_i is the h_i conjugate of $X_1 \to U_1$. If $\bar{X}_1$ is the closed fiber of $X_1 \to U_1$, then $\bar{X}_1 = \bar{X} \times_k k(\rho)$. It is immediate from the above discussion that if $E_i \subset X$ is a limb, then its preimage in X' is a disjoint union of limbs. We can number things so that the unique maximal of R'^h lies over M_1.

In the special case $\rho \in k$ or $\bar{H} = 1$, then $R^h = R'^h$. The maximal ideals of R' form a single free orbit under the H action. Furthermore, $\bar{X}_1 \cong \bar{X}$. If $\pi^h \in R^h$ is a purely inseparable prime, then the degree of ρ over R^h/π^h is the same as the degree of ρ over k_π which is the degree of ρ over k and hence $R' = R(\rho)$ over R. It follows that π^h has a unique extension to a prime π'^h of R'^h.

We are ready to state some properties of this set-up. With π as above, let $\pi_1, \dots, \pi_s$ be the extensions of π to R'. Let $C \subset X$ be the strict transform of $\{\pi = 0\}$ and $C_1, \dots, C_s \subset X'$ the strict transforms of $\{\pi_i = 0\}$. Note that C_i may map to more than one closed point of $\operatorname{Spec}(R')$.

Lemma 5.12. *Assume C is nonsingular, meets each exceptional line of $X \to \operatorname{Spec}(R)$ in at most one point, and each point of C is only on one exceptional line.*

(a) *The C_i are nonsingular and disjoint and meet each exceptional line of $X' \to \operatorname{Spec}(R')$ in at most one point, and each point of C_i is on only one exceptional line.*
(b) *Let $P' \in X'$ be a point on C_1. There is a unipoint prime η'_1 of R' such that if $D'_1 \subset X'$ is the strict transform of $\eta'_1 = 0$ then D'_1 contains and is nonsingular at P' with tangent distinct from that of C_1 and D'_1 contains no other point of X'.*
(c) *Let $E'_1 \subset X'$ be an exceptional line containing P', and suppose E'_1 is unique. Let Q' be another point of E'_1, not on C_1, on no other exceptional line, and with the same residue field as P'. There is a unipoint prime $\eta'_2 \in R'$ whose strict transform $D'_2 \subset X'$ contains and is nonsingular at Q' with tangent distinct from E'_1. $D'_1 - D'_2$ is the divisor of η'_1/η'_2 in X'.*
(d) *Let E'_1, $E'_2 \subset X'$ be two exceptional lines containing P'. Let $Q'_1 \in E'_1$ and $Q'_2 \in E'_2$ be other points, with the same residue field as P', not on C_1, and on no other exceptional line. There are unipoint primes $\eta'_2, \eta'_3 \in R'$ with strict transforms D'_2, D'_3 such that the D'_i contains and is nonsingular at Q'_i, have tangent distinct from the E'_i, and $D'_1 - D'_2 - D'_3$ is the divisor in X' of $\eta'_1/(\eta'_2\eta'_3)$.*

Proof. Part (a) follows because $X' \times_X C$ is etale over C. We turn to (b), (c) and (d). Note that if the unique point in $\operatorname{Spec}(R)$ does not split in $\operatorname{Spec}(R')$, then (b), (c) and (d) follow from 3.8(a) and 2.3. The main issue is to find the η'_1 in (b), the η'_2 in (c), and the η'_i in (d) which are as needed on $X_1 \to U_1$ and trivial on the others. We show how to do this in (b) as the other case are exactly parallel. In (c) and (d), note that 2.3 as applied to U_1 shows the that η'_1/η'_2 or $\eta'_1/(\eta'_2\eta'_3)$ define the divisors we claim.

Since R'/R is etale, $MR' = J' = M_1 \cap \cdots \cap M_s$ is the Jacobson radical of R'. More so, if $N > 0$, $R'/J^N \cong \bigoplus_i R'/M_i'^N$. If $\operatorname{Spec}(R'_i) = U_i$ as above, then $R'/M_i^N \cong R'_i/M_i^N$. By 3.8 we can find an η_1 as needed with respect to $X_1 \to \operatorname{Spec}(R'_1)$. By 3.5 there is an integer N such that if η'_1 is congruent to η_1 modulo M_1^N, then η'_1 is also as needed with respect to $X_1 \to \operatorname{Spec}(R'_1)$.

Now choose $\eta_1' \in R'$ congruent to η_1 modulo M_1^N and congruent to 1 modulo M_j^N for $j \neq 1$. Then η_1' is as needed. □

Let us use 5.12(b) to reduce 5.10 to a simpler case.

Lemma 5.13. *In order to prove* 5.10*, we may assume* π *is a unipoint prime of* R *which splits completely in* $R(\rho)$.

Proof. Since $\rho \in R^h$, we can assume $R \subset R' \subset R^h$. Let $\pi' = \pi_1 \in R'$ be the prime over π defined by π^h. $\bar{R}'_{\pi'}$ is the integral closure of R'/π', so $\bar{R}_\pi \subset \bar{R}'_{\pi'} \subset \bar{R}^h_{\pi^h}$. Let $P' \subset \bar{R}'_{\pi'}$ be the maximal over P defined by $\bar{R}^h_{\pi^h}$. In the notation of 5.12, π' corresponds to C_1 and P' to a point on C_1 as in 5.12(b). Let $\gamma' = \eta_1'/\eta_2'$ or $\gamma' = \eta_1'/(\eta_2'\eta_3')$ be as in 5.12(c) or (d) respectively, depending on whether P' is on one or two exceptional lines. Let $\tilde{\gamma}'$ be the image of γ' in $q(R'/\pi')$. Then $\tilde{\gamma}'$ has valuation 1 with respect to P' and is a unit with respect to all the other prime ideals of $\bar{R}'_{\pi'}$. Consider $L' = q(R'/\pi')(\tilde{\gamma}'^{1/n})$. By 5.8 the corestriction of $L'/q(R'/\pi')$ has the ramification required at π. Form the symbol algebra class $\alpha' = (\gamma', \pi')_{n,q(R')}$. Then α' ramifies only at π' and the unipoint primes η_i'. Let η_i be the restrictions of the η_i' to R. Clearly the η_i are unipoint, and since H has no stabilizer on P', the η_i split completely in R'. Since $\bar{R}_\pi/P$ is k_π, all the primes η_i are purely inseparable.

Let $\alpha^* \in \mathrm{Br}(q(R))$ be the corestriction of α'. Then α^* ramifies only at π and the η_i. By our assumption and 5.11, there are α_i which ramify only at η_i and δ, and whose ramification at η_i is the inverse of that of α^*. Then setting $\alpha = \alpha^*\alpha_1\alpha_2$ or $\alpha = \alpha^*\alpha_1\alpha_2\alpha_3$, 5.13 is proven. □

We are ready to prove 5.10. Of course, we have that π^h is purely inseparable and lies over a unipoint prime $\pi \in R$, meaning we can identify $\pi = \pi^h$. Furthermore, we can assume π splits completely in R' and that n is a prime power. Let $X \to \mathrm{Spec}(R)$ be a sequence of blow-ups of minimal length such that the strict transform, C, of $\pi = 0$ is nonsingular. By assumption C has a unique point P and C splits completely in $X' = X \times_R R'$. Let $\phi : X \to X_1$ be the last blow-up in the sequence with exceptional line E. By minimality $P \in E$. For the same reason, if $P_1 \in X_1$ and $C_1 \subset X_1$ are the images of P, C, then the multiplicity of C_1 through P_1 must be larger than 1. Since this is the multiplicity of E in $\phi^*(C_1)$, it follows from e.g. 2.2 that C and E intersect with multiplicity bigger than 1. That is, they have the same tangent at P. As in the proof of 5.13, there are two cases, depending on whether P is on one or two exceptional lines. We do the first case, as the second is similar.

As before we can assume $R' \subset R^h$, and take δ to be a nonsingular prime of R and hence R' and R^h. We take π' to be the prime of R' induced by π as a prime of R^h. Let P' be the point induced by $\bar{R}^h_\pi$ on the exceptional line E_1'. By 5.12 there is a unipoint $D' \subset X'$ defining $\eta' \in R'$ such that D' contains and is nonsingular at P' and has tangent different from E_1'. Again, there is a point $Q' \neq P'$, $Q' \in E_1'$, with the same residue field and not on any other exceptional line. Further, there is a unipoint $D^{*\prime} \subset X'$ containing and nonsingular at Q', with tangent different from E_1', and such that $D' - D^{*\prime}$ is the divisor of $\gamma' = \eta'/\eta^{*\prime}$. Just as in the proof of 5.13, we can let α' be the Brauer class of $(\gamma', \pi')_n$ in $\mathrm{Br}(q(R'))$. Then α' ramifies only at π, η', η'', and the ramification of α' at π itself has ramification ρ. Set $\alpha = \mathrm{Cor}_{q(R)}^{q(R')}(\alpha')$ and let η, η^*, be the restrictions of η' and $\eta^{*\prime}$ respectively to R. The same computation as in 5.13 shows that proving 5.10 for π reduces to proving it for η' and $\eta^{*\prime}$.

Let $X' \to X'_1$ be the base change to $\mathrm{Spec}(R')$ of $X \to X_1$. If $D'_1, D_1^{*\prime} \subset X'_1$ are the images of D' and $D^{*\prime}$, then D'_1 and D''_1 are already nonsingular, as are their images $D_1, D_1^* \subset X_1$, so the strict transform of $\eta = 0$ and $\eta^* = 0$ is already nonsingular in X_1.

By induction on the number of blow-ups in $X \to \mathrm{Spec}(R)$, we have reduced to the case that π itself is nonsingular. If π and δ have distinct tangents then the corestriction of the symbol algebra $(\pi', \delta)_n$ yields the needed Brauer class. If not, we introduce a third nonsingular prime ζ, with distinct tangent, and consider the Brauer class induced by the corestriction of $(\pi/\delta, \zeta)_n$. □

To sum up, we have proven 5.10 which yields 5.3. It is time to tackle 5.1 and 5.2. We have the diagram:

$$\begin{array}{ccccc}
\mathrm{Br}(q(R)) & \longrightarrow & \bigoplus_\pi H^1(q(R/\pi), \mathbb{Q}/\mathbb{Z}) & \rightarrow & \mu^{-1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Br}(q(R^h)) & \longrightarrow & \bigoplus_{\pi^h} H^1(q(R^h/\pi^h), \mathbb{Q}/\mathbb{Z}) & \rightarrow & \mu^{-1}.
\end{array}$$

The next key result is:

Proposition 5.14. *Suppose $\gamma^h \in \bigoplus_{\pi^h} H^1(q(R^h/\pi^h), \mathbb{Q}/\mathbb{Z})$ maps to 1 in μ^{-1} and is the image of $\gamma \in \bigoplus_\pi H^1(q(R/\pi), \mathbb{Q}/\mathbb{Z})$. Then there is an $\alpha \in \mathrm{Br}(q(R))$ whose image $\alpha^h \in \mathrm{Br}(q(R^h))$ has ramification γ^h.*

Proof. By 5.3 we can assume all the components of γ^h are unramified. Let $\pi \in R$ be a prime and $L/q(R/\pi), \sigma$ the component of γ at π. Then the prime ideals of $\bar{R}_\pi$ correspond to the factors of $\pi = \pi_1, \ldots, \pi_s$ of π in R^h and each $\bar{R}^h_{\pi_i}$ is the henselization of $\bar{R}_\pi$ localized at the corresponding prime ideal. Thus $L/q(R/\pi)$ is unramified at each of the prime ideals of $\bar{R}_\pi$, implying it comes from an extension of $\bar{R}_\pi$. Thus 5.14 follows from 4.1. □

We are ready to prove 5.1 and 5.2. We begin with the later. Suppose $\gamma \in \bigoplus_\pi H^1(q(R/\pi), \mathbb{Q}/\mathbb{Z})$ maps to 1 in μ^{-1}. Then the image

$$\gamma^h \in \bigoplus_{\pi^h} H^1\big(q\big(R^h/\pi^h\big), \mathbb{Q}/\mathbb{Z}\big)$$

is the image of some $\alpha \in \mathrm{Br}(q(R))$. That is, we may assume $\gamma^h = 0$. But then all the components of γ are unramified and we are done by 4.1.

Turning to 5.1, let $\alpha^h \in \mathrm{Br}(q(R^h))$ map to $\gamma^h \in \bigoplus_{\pi^h} H^1(q(R^h/\pi^h), \mathbb{Q}/\mathbb{Z})$. As a first step, we show that γ^h is the image of some $\gamma \in \bigoplus_\pi H^1(q(R/\pi), \mathbb{Q}/\mathbb{Z})$. By 5.3 we may assume all the nonzero components of γ^h are unramified, or occur at a single nonsingular prime—which also implies that component is unramified.

Lemma 5.15. *Let n be prime to the characteristic of k and let $\pi \in R$ be a prime which factors into primes $\pi_1, \ldots, \pi_s$ of R^h. Let $L_i/q(R^h/\pi_i), \sigma_i$ be degree n unramified cyclic extensions. Then under the hypotheses of* 5.1 *there is a degree n $L/q(R/\pi), \sigma$ which maps to all of them.*

Proof. Since the $\bar{R}^h_{\pi_i}$ are henselian, the L_i are determined by their residue extensions l_i/k_{π_i}. Thus 5.15 is equivalent to finding $L/q(R/\pi)$ with these residue extensions, and this is 4.5. $\square$

Now 5.1 is immediate from 5.14.

6. Genuine surfaces

We can view Section 1 as the study of the Brauer group of $q(R)$ where R is a henselian regular local surface. Section 5 studies the Brauer group of $q(R)$ for R regular local dimension 2 but not necessarily henselian. In this section we want to describe the Brauer group of K where K is the field of fractions of a general surface. Some of this material is known but hard to find in print. Much of it can be thought to be an extension of the discussion in [M, pp. 106–110]. We attempt, for the sake of the reader, to stick to etale cohomology as developed in that book.

To proceed, let S be a nonsingular surface with field of fractions $K = F(S)$. By this we mean a two dimensional excellent separated nonsingular integral Noetherian scheme projective over some affine scheme. A curve on S will mean a codimension 1 subscheme and a point on S will be a codimension 2 subscheme. However, "generic point" will have its usual meaning. If $\mathcal{F}$ is a torsion etale sheaf of X then $\mathcal{F}'$ will be the subsheaf of elements of order prime to all characteristics. Note that $H^0(X, \mathcal{F}') = H^0(X, \mathcal{F})'$.

For every curve $C \subset S$, the stalk $R_C = \mathcal{O}_{S,C}$ defines a discrete valuation domain and hence a ramification map $0 \to \mathrm{Br}(R_C)' \to \mathrm{Br}(K)' \to H^1(F(C), \mathbb{Q}/\mathbb{Z})' \to 0$ where $H^1(F(C), \mathbb{Q}/\mathbb{Z}) = H^1(G_{F(C)}, \mathbb{Q}/\mathbb{Z})$ and $G_{F(C)}$ is the absolute Galois group of the residue field $F(C)$. By e.g. [S1, p. 30] we have that $\mathrm{Br}(S) = \bigcap_{C \subset S} \mathrm{Br}(R_C)$ the intersection being over all curves on S. Thus there is an exact sequence

$$0 \to \mathrm{Br}(S)' \to \mathrm{Br}(K)' \to \bigoplus_{C \subset S} H^1\big(F(C), \mathbb{Q}/\mathbb{Z}\big)'$$

and our goal is to study the cokernel of the rightmost map.

In Section 1 we saw that when $S = \mathrm{Spec}(R)$ for R regular local, there is a map r: $\bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})' \to \mu^{-1}$ and in Section 5 we showed that $\mathrm{Br}(K)' \to \bigoplus_{C \subset S} H^1(F(C), \mathbb{Q}/\mathbb{Z})' \to \mu^{-1}$ is exact in this case. On this slim ground we consider the local case as "known" and try and use it to get at the general case. To some extent all we will need is the henselian case. We proceed by applying etale cohomology and the following exact sequence.

Let S be as above, and G_m the sheaf of units. Let $g : \mathrm{Spec}(K) \to S$ the generic point and $g_* G_{m,K}$ the push forward of the units sheaf on $\mathrm{Spec}(K)$. For each curve $C \subset S$, let $\iota_C : \mathrm{Spec}(F(C)) \to S$ be the generic point of C and $\iota_{C*}\mathbb{Z}$ the pushforward of the constant sheaf $\mathbb{Z}$. There is an exact sequence $0 \to G_m \to g_* G_{m,K} \to \bigoplus_{C \subset S} i_{C*}\mathbb{Z} \to 0$ of etale sheaves (e.g. [M, p. 72]). This yields the exact sequence

$$H^2(S, G_m) \to H^2(S, g_* G_{m,K}) \to \bigoplus_{C \subset S} H^2(S, \iota_{C*}\mathbb{Z})$$
$$\to H^3(S, G_m) \to H^3(S, g_* G_{m,K})$$

which we investigate.

To consider this sequence we need to tackle cohomology of the form $H^2(S, f_*\mathcal{F})$ where $f : X \to S$ and $\mathcal{F}$ is an etale sheaf on X. To study these we use the Leray spectral sequence

$$H^p\big(S, R^q f_*\mathcal{F}\big) \quad \Rightarrow \quad H^{p+q}(X, \mathcal{F})$$

in the case $p + q = 2$ and f is either $g : \mathrm{Spec}(K) \to S$ or $\iota_C : \mathrm{Spec}(F(C)) \to S$. Since $R^0 f_* = f_*$ one term of the spectral sequence appears in the sequence above. To understand the higher derived sheaves we recall the following. Suppose p is a geometric point of S corresponding to a separable closure. Let $i : \mathrm{Spec}(R_p) \to S$ be a stalk in the etale topology, meaning that R_p is the strict henselization of $\mathcal{O}_{S,q}$ for q the image of p. For easy reference we quote the standard result (e.g. [M, p. 88]).

Lemma 6.1. *The stalk of $R^i f_*\mathcal{F}$ at p is $H^i(X \times_S \mathrm{Spec}(R_p), \pi^*\mathcal{F})$ where $\pi : X \times_S \mathrm{Spec}(R_p) \to X$ is the projection.*

With the above lemma in mind let us describe $X \times_S \mathrm{Spec}(R_p)$ in all of our cases.

Lemma 6.2. *Suppose* $\mathrm{Spec}(R_p) \to S$ *is as above.*

(a) *Let $g : \mathrm{Spec}(K) \to S$ be as above. Then $\mathrm{Spec}(K) \times_S \mathrm{Spec}(R_p) = \mathrm{Spec}(K_p)$, where $K_p = q(R_p)$ is the field of fractions of R_p. In the notation of the above lemma, $\pi^*(G_{m,K}) = G_{m,K_p}$.*

Suppose $C \subset S$ is a curve and $\iota_C : \mathrm{Spec}(F(C)) \to S$ is as above.

(b) *If q is not on C, $\mathrm{Spec}(F(C)) \times_S \mathrm{Spec}(R_p)$ is the empty set.*
(c) *If q is the generic point on C, $\mathrm{Spec}(F(C)) \times_S \mathrm{Spec}(R_p) = \mathrm{Spec}(F(p))$ where $F(p)$ is the residue field of R_p and hence separably closed.*
(d) *If q is a closed point on C, $\mathrm{Spec}(F(C)) \times_S \mathrm{Spec}(R_p) = \mathrm{Spec}(F(p, C))$ where $F(p, C)$ is as follows. Let $P \subset \mathcal{O}_{S,q}$ correspond to C and let $P_1, \ldots, P_k$ be the prime ideals of R_p lying over P. Then $F(p, C) = \bigoplus_i q(R_p/P_i)$. In this case $\pi^*\mathbb{Z}$ is the constant sheaf $\mathbb{Z}$.*

Proof. (a) R_p is the direct limit of etale neighborhoods T and $\mathrm{Spec}(K) \times_S \mathrm{Spec}(T) = \mathrm{Spec}(K \otimes_{\mathcal{O}_{S,q}} T) = \mathrm{Spec}(q(T))$. The first statement follows by taking direct limits. The open subsets of $\mathrm{Spec}(K)$ are the set of $\mathrm{Spec}(L)$ for L/K finite separable and so the second sentence follows from the description of π^*.

Part (b) is clear, and (c) follows because if $R = \mathcal{O}_{S,C}$ has maximal ideal P, $F(C) = R/P$ and $R/P \otimes T = T/PT$ for T/R etale. But etale implies unramified.

As for (d), let T be an etale neighborhood of $\mathcal{O}_{S,q}$. Then $F(C) \otimes T$ is the total ring of fractions of $R/P \otimes T = T/PT$. The only thing left to observe is that there are only finitely many prime ideals of R_p lying over P, which follows because R_p is Noetherian. □

One use of the above lemma is to show the standard results:

Lemma 6.3.

(a) $R^1 g_* G_{m,K} = R^1 \iota_{C*}\mathbb{Z} = 0.$

(b) *There are exact sequences*

$$0 \to H^2(S, g_*G_{m,K}) \to H^2(K, G_m) \to H^0\big(R^2 g_* G_{m,K}\big)$$

and

$$0 \to H^2\Bigg(S, \bigoplus_C \iota_{C*}\mathbb{Z}\Bigg) \to \bigoplus_C H^2\big(F(C), \mathbb{Z}\big) \to \bigoplus_C H^0\big(S, R^2\iota_{C*}\mathbb{Z}\big).$$

(c) *$H^3(S, g_*G_{m,K})$ and $H^3(S, G_m)$ are torsion groups.*

Proof. In (a), the first is an etale sheaf whose stalk at any geometric point p is $H^1(q(R_p), G_m)$ which is 0 by Hilbert 90. For the second sheaf, the cohomology at the empty set or at a separable closed field must be 0, so the only real case is where p lies over a closed point q on the curve C. By (d), the stalk of $R^1\iota_{C*}\mathbb{Z}$ is $H^1(F(p,C), \mathbb{Z}) = 0$ since $F(p,C)$ is a finite direct sum of fields, and for a field F', $H^1(F', \mathbb{Z}) = \mathrm{Hom}(G_{F'}, \mathbb{Z}) = 0$ since $\mathbb{Z}$ has no torsion. Part (b) follows from (a) and the spectral sequence.

As for (c), we begin with the first fact. We use the spectral sequence

$$H^p\big(S, R^p g_* G_{m,K}\big) \Rightarrow H^{p+q}(K, G_m).$$

Since $H^3(K, G_m)$ is torsion, we need only look at the kernel of $H^3(S, g_*G_{m,K}) \to H^3(K, G_m)$. But this kernel is the image of $H^0(S, R^2 g_* G_{m,K})$ so it suffices to show that $R^2 g_* G_{m,K}$ is torsion. But this is the sheafification of a presheaf of the form $U \to \mathrm{Br}(F(U))$ making this clear. The kernel of $H^3(S, G_m) \to H^3(S, g_*G_{m,K})$ is the image of $\bigoplus_C H^2(S, \iota_{C*}\mathbb{Z})$. This later group embeds in $\bigoplus_C H^2(K, \mathbb{Z}) \cong \bigoplus_C H^1(K, \mathbb{Q}/\mathbb{Z})$ and so it is clearly torsion. □

We can further describe some of the cohomology groups in (b) above, once we restrict to prime to the characteristic parts. From the sheaf property and 6.2 and 6.3 there is an injection $H^0(S, g_*G_{m,K})' \to \prod_p H^2(R_p \times K, G_m)'$ where the product is over all the stalks. Note that the stalk at the codimension 0 point is a separably closed field, and if p lies over a codimension one point then K_p is the field of fractions of a strictly henselian discrete valuation domain and in both cases the cohomology (Brauer) group is 0. Thus the product above can be taken over all strict henselizations at closed points. This is part (a) of:

Lemma 6.4.

(a) *The maps to stalks defines an injection*

$$H^0\big(S, R^2 g_* G_{m,K}\big)' \to \prod_p H^2(K_p, G_m)'$$

the product being over all strict henselizations R_p at closed points and K_p being the field of fractions of R_p.

(b) *The map to stalks defines an injection*

$$H^0\big(S, R^2\iota_{C*}\mathbb{Z}\big)' \to \prod_{p \to C} H^2\big(F(p,C), \mathbb{Z}\big)'$$

where the product is over all stalks over closed points of C and $F(p,C)$ is as in 6.2(d).

Proof. To show (b), we must show we can ignore all other stalks which is clear from 6.2. □

We need to link the derived functors $R^2 g_* G_{m,K}$ and the $R^2 \iota_{C*}\mathbb{Z}$. To state the result, let R_p be a stalk at a closed point q on C. Recall that $F(p,C) = \bigoplus_i F(C_i)$ where C_i correspond to the prime ideals $P_i \subset R_p$ over P defining C. There is a map $r'_{p,C} : H^2(K_p, G_m) \to H^2(F(p,C),\mathbb{Z}) = \bigoplus_i H^2(F(C_i),\mathbb{Z})$ defined as the sum of the maps induced by the valuation associated to each C_i. If C does not contain the image of p we define $r'_{p,C} = 0$. Fixing C but varying p, we have $r'_C : \prod_p H^2(K_p, G_m) \to \prod_{p\in C} H^2(F(p,C),\mathbb{Z})$ whose p component is $r'_{p,C}$.

Proposition 6.5. *There is a natural sheaf morphism $\psi : (R^2 g_* G_{m,K})' \to \bigoplus_C R^2 \iota_{C*}\mathbb{Z}$ such that the induced diagram below commutes.*

$$\begin{array}{ccccc}
H^2(K,G_m)' & \longrightarrow & H^0(S, R^2 g_* G_{m,K})' & \longrightarrow & \prod_p H^2(K_p,G_m)' \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_C H^2(F(C),\mathbb{Z})' & \longrightarrow & \bigoplus_C H^0(S, R^2 \iota_{C*}\mathbb{Z})' & \longrightarrow & \bigoplus_C (\prod_{p\to C} H^2(F(p,C),\mathbb{Z})').
\end{array}$$

Furthermore, the left vertical map is sum of the maps induced by the valuation associated to each C and the right vertical map is the sum of r'_C's.

Proof. It is enough to prove this for a single C. Note that $g : \mathrm{Spec}(K) \to S$ and $\iota_C : \mathrm{Spec}(F(C)) \to S$ factor through $h : \mathrm{Spec}(\mathcal{O}_{S,C}) \to S$. Write $g = h \circ g'$, $\iota_C = h \circ \iota'$ and $R = \mathcal{O}_{S,C}$ a discrete valuation ring. Let $\hat{R}$ be the strict henselization of R, which is one stalk of $\mathrm{Spec}(R)$ while the separable closure $K_s \supset K$ is the other. Note that for $i > 0$, $R^i \iota'_* \mathbb{Z} = 0$ because the cohomology at both stalks is clearly 0. Furthermore, $R^1 g'_* G_{m,K} = 0$ as before while $R^2 g'_* G_{m,K}$ embeds in $H^2(q(\hat{R}), G_m)$ and $H^2(q(\hat{R}), G_m)' = 0$. This implies $R^2 g'_* G'_m = 0$. Using the spectral sequence we have that $R^2 \iota_{C*}\mathbb{Z} = R^2 h_*(\iota'_{C*}\mathbb{Z})$ and $R^2 g_* G'_{m,K} = R^2 h_*(g'_* G_{m,K})$. Thus to define ψ it is enough to define $\psi' : g'_* G_{m,K} \to \iota'_{C*}\mathbb{Z}$ which is just the valuation. That the vertical maps are as claimed is an easy exercise. □

Since $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is exact, and $H^i(F,\mathbb{Q}) = 0$ for $i > 0$ and any field F, we have $H^2(F(C),\mathbb{Z}) = H^1(F(C),\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_c(G_{F(C)},\mathbb{Q}/\mathbb{Z})$ and similarly for $H^2(F(C_i),\mathbb{Z})$. We frequently identify $H^2(F,\mathbb{Z})$ and $H^1(F,\mathbb{Q}/\mathbb{Z})$.

It is well known that $\mathrm{Br}(S)$ embeds in $\mathrm{Br}(K)$ (e.g. [M, p. 145]). Furthermore, $\mathrm{Br}(S)' = \bigcap_C \mathrm{Br}(\mathcal{O}_{S,C})'$ by e.g. [S1, p. 30]. Since $0 \to \mathrm{Br}(\mathcal{O}_{S,C})' \to \mathrm{Br}(K)' \to H^1(F(C),\mathbb{Q}/\mathbb{Z})' \to 0$ is exact (e.g., [AB, p. 289]) we have:

Lemma 6.6.

(a) *The sequence $0 \to \mathrm{Br}(S)' \to \mathrm{Br}(K)' \to \bigoplus_C H^1(F(C),\mathbb{Q}/\mathbb{Z})'$ is exact.*
(b) *The sheaf map $(R^2 g_* G_{m,K})' \to \bigoplus_C R^2 \iota_{C*}\mathbb{Z}$ is injective.*

Proof. In (b), we look at this map on stalks where it is just

$$H^2(K_p, G_m)' \to \bigoplus_C H^2\bigl(F(p,C), \mathbb{Z}\bigr).$$

Since every height one prime ideal of R_p lies over some C, the kernel of this map is $\mathrm{Br}(R_p)'$ which is 0 (e.g. [Se, p. 194]). $\square$

Just to recall what it means we write $F(p,C) = \bigoplus_{C_i/C} F(C_i)$. Further recall that $K_p = q(R_p)$ is the field of fractions of the strict henselization. Let $\prod_p$ represent the product over all points of S, and let $\prod_{p\to C}$ represent the product over all points of S on the curve C. Define Γ to be the cokernel of $H^2(K, G_m) \to \bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})$ and Δ the cokernel of

$$\psi : \prod_p \mathrm{Br}\bigl(q(R_p)\bigr) \to \bigoplus_C \biggl(\prod_{p\to C} \bigoplus_{C_i|C} H^1\bigl(F(C_i), \mathbb{Q}/\mathbb{Z}\bigr)\biggr).$$

Finally, for any field F we write $H^2(F, G_m) = \mathrm{Br}(F)$. Altogether we have the diagram:

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
H^2(S, g_*G_{m,K})' & \xrightarrow{f_1} & \bigoplus_C H^2(S, \iota_{C*}\mathbb{Z})' & \twoheadrightarrow & KH^3(S, G_m)' \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Br}(K)' & \xrightarrow{f_2} & \bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})' & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \\
H^0(S, R^2 g_*G_{m,K})' & \hookrightarrow & \bigoplus_C H^0(S, R^2\iota_{C*}\mathbb{Z})' & & \downarrow \\
\cap| & & \cap| & & \\
\prod_p \mathrm{Br}(K_p)' & \hookrightarrow & \bigoplus_C(\prod_{p\to C} \bigoplus_{C_i/C} H^1(F(C_i), \mathbb{Q}/\mathbb{Z}))' & \twoheadrightarrow & \Delta
\end{array}$$

where $KH^3(S, G_m)$ is the kernel of $H^3(S, G_m) \to H^3(S, g_*G_{m,K})$ and f_1, f_2 have kernel $\mathrm{Br}(S)'$. Our goal is to study Γ, but first we make some easy observations from this diagram. To state (b) below, let q be a closed point of a curve $C \subset S$ and $R = \mathcal{O}_{S,q}$ the Zariski stalk. Let $r_{C,q} : H^1(F(C), \mathbb{Q}/\mathbb{Z})' \to \mu^{-1}$ be the map r_P defined in Section 1, where $P \subset R$ is the prime ideal associated to C.

Lemma 6.7.

(a) $H^2(S, g_*G_{m,K})'$ *can be identified with the subgroup of* $\mathrm{Br}(K)'$ *which maps to* 0 *in all* $\mathrm{Br}(K_p)'$.

(b) *Let the subgroup A of $\bigoplus_C H^1(F(C),\mathbb{Q}/\mathbb{Z})'$ be defined by $(\chi_C)_C \in A$ if and only if $r_{C,q}(\chi_C)=0$ for all C and all closed points q on C. Then $H^2(S,\bigoplus_C \iota_{C*}\mathbb{Z})'$ can be identified with a subgroup of A.*

Proof. Part (a) is obvious. As for (b), let R_p be a stalk with residue field k_p and let $P_i \subset R_p$ correspond to C_i which lies over C, so $F(C_i)=q(R_p/P_i)$. Set $\bar{R}_i$ to be the integral closure of R_p/P_i with residue field k_i. Then k_p is separably closed so k_i/k_p is purely inseparable and hence has degree a power of the characteristic. In addition, $\bar{R}_i$ is strictly henselian and so all extensions are totally ramified. It follows that $r_{P_i}: H^1(F(C_i),\mathbb{Q}/\mathbb{Z})' \to \mu^{-1}$ is an isomorphism. Thus if $\chi_C \in H^1(F(C),\mathbb{Q}/\mathbb{Z})$ maps to 0 in $H^1(F(C_i),\mathbb{Q}/\mathbb{Z})$ for all $P_i \subset R_p$ and R_p lies over $\mathcal{O}_{S,q}$ it follows that $r_{q,C}(\chi_C)=0$ by 1.5.

Remark. It is clear from the proof of 6.7(b) that $H^2(S,\bigoplus_C \iota_{C*}\mathbb{Z})'$ can somehow be thought of as the χ_C which have $r_{q,C}(\chi_C)=0$ on all of the branches of C through all q.

We turn to our main goal of studying Γ.

Lemma 6.8. *The induced map $KH^3(S,G_m)' \to \Gamma$ is injective.*

Proof. Suppose $x \in KH^3(S,G_m)'$ maps to 0 in Γ. The element x is the image of $x' \in \bigoplus_C H^2(S,\iota_{C*}\mathbb{Z})$. Let x' map to $y \in \bigoplus_C H^1(F(C),\mathbb{Q}/\mathbb{Z})$. By assumption, y is the image of some $y' \in \mathrm{Br}(K)'$. Since y maps to 0 in $\bigoplus_C H^2(S,R^2\iota_{C*}\mathbb{Z})'$, we have that $y' \mapsto 0 \in H^0(R^2g_*G_{m,K})$, implying that y' is the image of $y'' \in H^2(S,g_*G_{m,K})'$. Since $y'' \mapsto x'$, we have $x=0$. $\square$

The next issue is the kernel of $\Gamma \to \Delta$ which we write as Θ. So suppose $x \in \Gamma$ maps to 0 in Δ. If $x' \in \bigoplus_C H^1(F(C),\mathbb{Q}/\mathbb{Z})'$ is a preimage of x, and $x' \mapsto y' \in \bigoplus_C H^0(S,R^2\iota_{C*}\mathbb{Z})'$, we claim y' is the image of $y \in H^0(S,R^2g_*G_{m,K})'$. Since x maps to 0 in Δ, it is clear that the images of y' in the stalks all come from the stalks of $R^2g_*G_{m,K}$. Let $\mathcal{F}$ be the sheaf quotient of $\bigoplus_C R^2\iota_{C*}\mathbb{Z}$ by the subsheaf $R^2g_*G_{m,K}$. Then if y' maps to $y'' \in \mathcal{F}(S)$, it is clear y'' has all 0 stalks and thus is 0, implying y exists as needed. The spectral sequence implies the exactness of $H^2(K,G_m) \to H^0(S,R^2g_*G_{m,K}) \to H^3(S,g_*G_{m,K})$ and we set $\phi(x)$ to be the image of y in $H^3(S,g_*G_{m,K})$.

Theorem 6.9. *ϕ is a well defined homomorphism from Θ to $H^3(S,g_*G_{m,K})'$ with kernel the image of $KH^3(S,G_m)'$.*

Proof. The ambiguity in the definition of ϕ comes in the choice of preimage x' of x. But any two such differ by the image of $x'' \in H^2(K,G_m)'$. Since x'' maps to 0 in $H^3(S,g_*G_{m,K})'$, this does not change $\phi(x)$. On the other hand, suppose $\phi(x)=0$, implying that y is the image of $x'' \in H^2(K,G_m)'$. Then the image of x'' and x' both map to y', and so $x'-x''$ is the image of $z \in \bigoplus_C H^2(S,\iota_{C*}\mathbb{Z})'$, and z maps to some $z' \in KH^3(S,G_m)'$. Since x'' maps to 0 in L, x is the image of z'. $\square$

We consider ϕ a bit more.

Proposition 6.10.

(a) *The image of ϕ is the intersection of the kernels of*

$$H^3(S, g_*G_{m,K})' \to H^3(K, G_m)'$$

and

$$H^3(S, g_*G_{m,K})' \to \bigoplus_C H^3(S, \iota_{C*}\mathbb{Z})'$$

which is the same as the intersection of the kernel of

$$H^3(S, g_*G_{m,K})' \to H^3(K, G_m)$$

and the image of $H^3(S, G_m)'$.

(b) *Let $U \subset S$ be open. The composition $\Theta \to H^3(S, g_*G_{m,K})' \to H^3(U, g_*G_{m,K})'$ is just the map ϕ_U defined using the surface U instead of S.*

Proof. We do the harder direction of (a), the other one being similar but easier. Suppose $\gamma \in H^3(S, g_*G_{m,K})'$ is in the intersection of these kernels. From the spectral sequence γ is the image of some $\gamma' \in H^0(S, R^2g_*G_{m,K})'$ and if γ' maps to $\gamma'' \in \bigoplus_C H^0(S, R^2\iota_{C*}\mathbb{Z})'$, then γ'' maps to 0 in $\bigoplus_C H^3(S, \iota_{C*}\mathbb{Z})'$. From the spectral sequence γ'' is the image of $\gamma''' \in \bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})'$, and the image of γ''' in Γ is the preimage of γ. Note that, from the long exact sequence, the image of $H^3(S, G_m)'$ is the kernel of $H^3(S, g_*G_{m,K})' \to H^3(S, \bigoplus_C \iota_{C*}\mathbb{Z})'$ Part (b) is immediate from naturality. $\square$

Thus to full study Γ we need to describe the map $\Gamma \to \Delta$. To make sense of this map we first study Δ. To begin with, we can rewrite

$$\bigoplus_C \prod_{p \to C} \bigoplus_{C_i/C} H^1\big(F(C_i), \mathbb{Q}/\mathbb{Z}\big)'$$

more simply as $\prod_p(\bigoplus_{C_p} H^1(F(C_p), \mathbb{Q}/\mathbb{Z})')$ where the C_p run over all the curves of $\mathrm{Spec}(R_p)$. The induced map $\psi : \prod_p \mathrm{Br}(K_p)' \to \prod_p(\bigoplus_{C_p} H^1(F(C_p), \mathbb{Q}/\mathbb{Z}))'$ is by 6.5 the product of the maps $\mathrm{ram}_p : \mathrm{Br}(q(R_p))' \to \bigoplus_{C_p} H^1(F(C_p), \mathbb{Q}/\mathbb{Z})'$. By 1.7 the cokernel of ram_p is $(\mu^{-1})'$. Thus Δ is the product of $(\mu^{-1})'$ over all R_p.

To understand the map $\Gamma \to \Delta$ we proceed as follows. Suppose an element of Γ has preimage $(\chi_C)_C \in \bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})'$. Let q be a closed point on $C_1, \ldots, C_r$ and R_p the strict henselization of $\mathcal{O}_{S,q}$. Let C_{ij} be the curves of $\mathrm{Spec}(R_p)$ lying over C_i. By 1.5 the composition

$$H^1\big(F(C), \mathbb{Q}/\mathbb{Z}\big)' \longrightarrow \bigoplus_{C_{ij}} H^1\big(F(C_{ij}), \mathbb{Q}/\mathbb{Z}\big)' \xrightarrow{\mathrm{ram}_p} \mu^{-1}$$

is just the map $r_{q,C}$ defined in Section 1. We form the direct sum over all closed points $q \in S$ so we have a map $r : \bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})' \to \bigoplus_q \mu^{-1}$ and we know by 1.1 that the composition $r \circ \mathrm{ram} : \mathrm{Br}(K)' \to \bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})' \to \bigoplus_q \mu^{-1}$ is the trivial map. The following is clear.

Proposition 6.11.

(a) *The map $\Gamma \to \Delta = \prod_p \mu^{-1}$ is induced by $(\chi_C)_C \mapsto (r_q) \in \prod_q \mu^{-1}$ where the product is over all closed points of S and $r_q = \sum_{C_i} r_{q,C_i}(\chi_{C_i})$, the sum being over all curves C_i of S containing q.*
(b) *Θ is the homology of*

$$\mathrm{Br}(K)' \xrightarrow{\mathrm{ram}} \bigoplus_C H^1\bigl(F(C), \mathbb{Q}/\mathbb{Z}\bigr)' \xrightarrow{r} \bigoplus_q \mu^{-1}.$$

By 6.9 we have an exact sequence $0 \to KH^3(S, G_m)' \to \Theta \to H^3(S, g_*G_{m,K})'$ and of course, by the definition, we have an exact sequence $0 \to KH^3(S, G_m)' \to H^3(S, G_m)' \to H^3(S, g_*(G_{m,K}))'$. This suggests Θ is closely related to $H^3(S, G_m)'$.

Theorem 6.12. *We can identify Θ with the kernel of $H^3(S, G_m)' \to H^3(K, G_m)'$. That is, if S is an excellent regular Noetherian surface of dimension 2 there is a sequence:*

$$0 \to \mathrm{Br}(S)' \to \mathrm{Br}(K)' \to \bigoplus_C H^1\bigl(F(C), \mathbb{Q}/\mathbb{Z}\bigr)' \to \bigoplus_q \mu^{-1}$$

which is exact at all places except $\bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})$ and there the homology is isomorphic to the kernel of $H^3(S, G_m)' \to H^3(K, G_m)'$.

Proof. We begin this proof with:

Lemma 6.13. *Suppose $U \subset S$ is open such that $Z = S - U$ is a finite union of closed points.*

(a) *$H^2(S, G_m)' \cong H^2(U, G_m)'$.*
(b) *$H^3(S, G_m)' \to H^3(U, G_m)'$ is injective.*

Proof. Starting with (a), both these groups are Brauer groups. Since S is regular, the restriction map is injective. By [S1, p. 30], an element of $\mathrm{Br}(U)$ has to ramify along codimension one curves, implying it is in the image of $\mathrm{Br}(S)$.

Turning to (b), suppose $\gamma \in H^3(S, G_m)'$ maps to 0 in $H^3(U, G_m)$. Let $n\gamma = 0$ for n prime to any characteristic. There is an exact sequence of etale sheaves

$$0 \to \mu_n \to G_m \xrightarrow{n} G_m \to 0$$

where n induces the same map on all cohomology. Thus γ is the image $\gamma' \in H^3(S, \mu_n)$. For any n', the kernel of $H^3(S, \mu_{n'}) \to H^3(U, \mu_{n'})$ is the image of $H^3_Z(S, \mu_{n'})$ [M, p. 92]. But by [M, p. 241] this group is 0, so $H^3(S, \mu_{n'}) \to H^3(U, \mu_{n'})$ is injective. Let $\gamma'_U \in H^3(U, \mu_n)$ be the image of γ', which must map to 0 in $H^3(U, G_m)$. That is, γ'_U is the image of some $\gamma'' \in H^2(U, G_m) = H^2(S, G_m)$. But γ'' must map to γ' showing that $\gamma = 0$. $\square$

Let us return to 6.12. We are going to perform this proof by restricting to open sets as in 6.13. Let us first note that if $g_U : \mathrm{Spec}(K) \to U$ is the generic point, and $\iota_{C,U} : \mathrm{Spec}(F(C)) \to U$ is the generic point of $U \cap C$, then $(g_U)_* G_{m,K}$ is the restriction of $g_* G_{m,K}$ to U and $\iota_{C,U*}\mathbb{Z}$ is the restriction of $\iota_{C*}\mathbb{Z}$ to U. It thus makes sense to write these restrictions as $g_* G_{m,K}$ and $\iota_{C*}\mathbb{Z}$.

We define $\Phi : \Theta \to H^3(S, G_m)'$ as follows. Suppose $\gamma \in \Theta$ is the image of $\gamma' = (\chi_C)_C \in \bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})'$. Let $C_1, \ldots, C_r$ be the curves where $\chi_{C_i} \neq 0$. By 6.11 the only points q where $r_{q,C}(\chi_C) \neq 0$ must be intersection points of the C_i and hence must be finite in number. If Z is that finite set of points, let $U = S - Z$. By 6.7 γ' is the image of some $\gamma'' \in H^2(U, \bigoplus_C \iota_{C*}\mathbb{Z})'$ which maps to $\gamma''' \in KH^3(U, G_m)'$.

Proposition 6.14. *γ''' is in the image of $H^3(S, G_m)'$.*

Proof. By induction we can assume $S - U$ is exactly one point z. There is an exact sequence $H^3(S, G_m) \to H^3(U, G_m) \to H^4_z(S, G_m)$ by [M, p. 92] and by [M, p. 93] $H^4_z(S, G_m) \cong H^4_z(\mathrm{Spec}(R_z), G_m)$ where R_z is the henselization of $\mathcal{O}_{S,z}$. Thus to prove this we may assume $S = \mathrm{Spec}(R_z)$. But for this S we know by 1.7 that $\Theta = 0$, implying the result. $\square$

Using 6.14 we define $\Phi(\gamma)$ to be the unique element of $H^3(S, G_m)'$ mapping to γ'''. Since Φ is the inverse of the injective map $KH^3(U, G_m)' \to \Gamma$ of 6.8 it is well defined and injective. For the rest, we would like to directly compare Φ and the map ϕ of 6.10. This is technically too difficult, but we can draw a small fact from 6.10 which turns out to be enough.

Lemma 6.15. *Suppose $\eta \in H^3(S, g_* G_{m,K})'$ is in the image of ϕ from* 6.10. *Then there is a finite set of points $Z \subset S$ such that if $U = S - Z$, η maps to* 0 *in $H^3(U, g_* G_{m,K})$.*

Proof. Suppose $\eta = \phi(\gamma)$ and γ is the image of $(\chi_C)_C \in \bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})$. As above, if C_i are the curves with $\chi_{C_i} \neq 0$ and Z is the intersection points of this curves, then $(\chi_C)_C$ is in the image of $H^2(U, \bigoplus_C \iota_{C*}\mathbb{Z})'$ and hence, in the notation of 6.10, $\phi_U(\gamma) = 0$ implying our result by naturality. $\square$

Now we can prove the surjectivity of 6.12. Suppose $\beta \in H^3(S, G_m)'$ maps to 0 in $H^3(K, G_m)'$. Let $\beta' \in H^3(S, g_* G_{m,K})'$ be its image. By 6.10 β' is in the image of ϕ. By 6.15 there is a $U = S - Z$ as in 6.15 such that β' maps to 0 in $H^3(U, g_* G_{m,K})$. If β'' is the image of β in $H^3(U, G_m)'$, then β'' is the image of some element $\gamma' \in H^2(U, \bigoplus_C \iota_{C*}\mathbb{Z})'$ we can regard as an element of $\bigoplus_C H^1(F(C), \mathbb{Q}/\mathbb{Z})'$. The image of γ' in Θ is our preimage and this proves 6.12.

Theorem 6.12 has a consequence whose significance is not yet clear. Note that here, and only here, we assume S is projective over a separably closed field.

Theorem 6.16. *Let S be a surface projective over a separably closed field. Let $\pi : S' \to S$ be the result of a sequence of blow-ups. Then the map $H^3(S, G_m) \to H^3(S', G_m)$ is an isomorphism when restricted to the elements going to* 0 *in $H^3(F(S), G_m)$.*

Proof. Let $U \subset S$ be the set such that $\pi : \pi^{-1}(U) \to U$ is an isomorphism, so $S - U$ is a finite set of points. Set $E = \pi^{-1}(S - U)$. We have the commutative diagram:

$$\begin{array}{ccc} H^3(S, G_m) & \longrightarrow & H^3(S', G_m) \\ \downarrow & & \downarrow \\ H^3(U, G_m) & \cong & H^3(S' - E, G_m) \end{array}$$

where the left vertical arrow is injective and the map on the bottom row is, of course, an isomorphism. It follows that $H^3(S, G_m) \to H^3(S', G_m)$ is injective.

For the surjectivity, we may assume $S' \to S$ is one blow-up and so $E \cong \mathbb{P}^1_k$, k being the residue field at the unique point $S - U$. Let $\alpha \in H^3(S', G_m)$ be represented by $(\chi_C)_C \in \bigoplus_{C \subset S'} H^1(F(C), \mathbb{Q}/\mathbb{Z})$. If $\chi_E \neq 0$, we can modify by a Brauer $F(S')$ element and assume $\chi_E = 0$. Thus we can view $(\chi_C)_C$ as in $\bigoplus_{C \subset S} H^1(F(C), \mathbb{Q}/\mathbb{Z})$. If this element over S has the sum of all ramifications at all points equal to 0, we are done. Of course, by assumption, this is true for S'. Thus to finish this result we prove:

Proposition 6.17. *Suppose R is a regular local ring of dimension 2 and residue field k. Let $P \subset R$ be a prime ideal corresponding to the curve $C \subset \operatorname{Spec}(R)$. Let $\alpha \in H^1(F(C), \mathbb{Q}/\mathbb{Z})$. Suppose $X \to \operatorname{Spec}(R)$ is the blow-up of R at the point and $Q_1, \ldots, Q_r$ are the intersection points of the strict transform of C and the exceptional divisor, and k_i is the residue field of X at Q_i. Then $r_P(\alpha) = \sum_i [k_i : k] r_{Q_i}(\alpha)$.*

Proof. Let $\bar{R}_P$ be the integral closure of R/P in $q(R/P)$. Suppose $R(u)$ represents an affine open subset $U \subset X$, where $uy = x$ and (x, y) is the maximal ideal of R. Another affine piece, which together cover, is $\operatorname{Spec}(R(v))$ where $vx = y$. The Q_i above correspond to maximal ideals of the semilocal local rings $(R/P)(\tilde{u})$ or $R/P(\tilde{v})$ or both. If R_i is the stalk of X at Q_i, and $P_i \subset R_i$ corresponds to the strict transform, then R_i/P_i is the localization of $(R/P)(\tilde{u})$ or $(R/P)(\tilde{v})$ at the corresponding maximal ideal. Looking at one affine piece, $(R/P)(\tilde{u}) \subset \bar{R}_P(\tilde{u})$ and the later is the localization of $\bar{R}_P$ at all the primes Q where $v_Q(x) \geqslant v_Q(y)$, and each of these Q lies over a unique maximal ideal of $(R/P)(\tilde{u})$. The remaining prime ideals lie over maximals of $(R/P)(\tilde{v})$. All together the prime ideals of $\bar{R}_P$ are partitioned among the Q_i, and the integral closure of R_i/Q_i is the localization of $\bar{R}_P$ at those prime ideals corresponding to Q_i. The result is now clear from the formula for r_P. □

7. Splitting ramification

Now we change direction a bit and consider the splitting of ramification for $\alpha \in \operatorname{Br}(K)$, where K is the fraction field of a surface S. In [S2] we showed that every prime degree $q \neq p$, a degree q division algebra D/K was cyclic, but only in the case S was proper over $\operatorname{Spec}(\mathbb{Z}_p)$ for the p-adic integers $\mathbb{Z}_p$.

The method used in [S2] involved showing that there was a degree q Kummer extension that split all the ramification. In this section we show just this fact, for very general S, but assuming S contains a primitive q root of one. We will make considerable use of the results and terminology of [S2], some of which we briefly review.

So let S be an excellent Noetherian regular surface of dimension 2 which is quasi-projective over an affine scheme. The key consequence of this assumption is that every finite set of points is contained in a single affine open subset. We further assume $\mathcal{O}_S$ contains a primitive q root of one, for q a prime unequal to any residue characteristic. Suppose $\alpha \in \operatorname{Br}(K)$ is of order q. If C_i is the full set of curves on S where α ramifies, and $L_i/F(C_i), \sigma_i$ is the ramification of α at C_i, we call the set of $L_i/F(C_i), \sigma_i$ the **ramification data** and the union of the C_i as the **ramification locus** of α. After blowing up we may assume that the C_i are nonsingular curves and intersect in normal crossings. Intersection points among the C_i are called **nodal** points. By assumption they lie on exactly two curves with distinct tangents. Points on exactly one curve are called **curve** points. By assumption they are nonsingular points on those curves.

We need to recall much more of the machinery of [S2]. There we classified the nodal points of the ramification locus into four categories—hot, cool, chilly, and cold. Let P be a nodal point at the intersection of C_1 and C_2. Recall that P is a **cold** point if one, and hence both, of the covers $L_i/F(C_i)$ are ramified at P. In the other three cases there are residue extensions $\bar{L}_i/F(P), \bar{\sigma_i}$ defined. P is a **cool** point if both covers split at P. P is a **hot** point if as extensions $\bar{L}_1/F(P)$ and $\bar{L}_2/F(P)$ are distinct. Finally P is a chilly point if $\bar{L}_1/F(P) = \bar{L}_2/F(P)$ and $\bar{\sigma_1}^{s_P} = \bar{\sigma_2}$. s_P is called the coefficient of P with respect to C_1. We showed that if α has a hot point, it cannot have index q and so we assume there are no hot points. We showed that after a blow up, we could eliminate cool points and loops of curves all nodes of which were chilly—so-called chilly loops. We assume this has been done.

The goal of this section is to find a $\pi \in K$ such that $K(\pi^{1/q})$ splits all the ramification of α. As we proceed, we will successively refine our choice of π until we are done. To begin with, let s_i be the order of π at C_i. If all the s_i are prime to q, then $K' = K(\pi^{1/q})$ kills all the ramification of α at all the C_i. From now on we assume this about π. As observed in [S2], this implies that we can define $\beta_{C_i} \in \mathrm{Br}(F(C_i))$, called a **residual class**, as follows. If $R_C = \mathcal{O}_{S,C}$ is the stalk at a curve C among the C_i, and since π has order prime to q at C, then $K' = K(\pi^{1/q})$ is totally ramified at C and it follows that R'_C, the integral closure of R_C in K', is a discrete valuation ring with residue field $F(C)$. Moreover, since the ramification of α at C has been split, α maps to an element of $\mathrm{Br}(R'_C)$ which maps to $\beta_C \in \mathrm{Br}(F(C))$. Finally we note [S2] that if $L/F(C_i)$ does not split β_{C_i}, then α does not have index q and we therefore assume $L_i/F(C_i)$ does split β_{C_i} whenever defined, saying thereby that α is **residually split**.

We quote from [S2]:

Theorem 7.1. *Suppose α is as above and the divisor (π) is equal to $\sum_i s_i C_i + E$ where all the s_i are prime to q, and the support of E does not contain any C_i or any nodal points. Let P be an intersection point of C_i and C_j with coefficient s_P with respect to C_i. Set $K' = K(\pi^{1/q})$.*

(a) *Suppose P is a chilly point. Then K' splits all the ramification of α over P if and only if $s_P = s_j(s_i)^{-1}$ in $(\mathbb{Z}/q\mathbb{Z})^*$.*
(b) *Suppose P is a cold point. Then K' splits all the ramification of α at P if and only if β_{C_i} is unramified at P.*

In fact we can show the following, which we quote from [S2].

Theorem 7.2. *Let α, C_i be as above. Elements $s_i \in (\mathbb{Z}/q\mathbb{Z})^*$ can be chosen such that for any chilly point P on the intersection of C_i and C_j, $s_P = s_j(s_i)^{-1}$ where s_P is the coefficient with respect to C_i. Furthermore, there is a $\pi \in K$, such that π has valuation s_i at the C_i, $(\pi) - \sum_i s_i C_i$ does not contain any nodal points or C_i in its support, and with respect to $K' = K(\pi^{1/q})$, all of the residual Brauer classes β_{C_i} are trivial. In particular, K' splits all the ramification of α at the C_i and at all the nodal points.*

Henceforth (until we change it) we will assume $(\pi) = \sum_i s_i C_i + E$ has the properties specified in 7.2.

This as far as we got in [S2] in splitting all ramification for general S. The remaining difficulty involves points on the intersection of E and the C_i. One can think of this as (π) "biting back."

To continue, let R be a regular local two dimensional domain with maximal ideal M and $\delta \in R$ a prime element. Then $R/\delta R$ is a one dimensional domain with integral closure we denote $\bar{R}_\delta$.

Let $F(\bar{\delta})$ be $\bar{R}_\delta/J$ where J is the Jacobson radical of $\bar{R}_\delta$. Thus $F(\bar{\delta})$ is a direct sum of fields all containing $R/M = k$. If $k' \supset k$ is an extension field, we say $F(\bar{\delta})$ **contains** k' if all the direct summands contain k'. Let C be a curve along which α ramifies with ramification $\tilde{L}/F(C), \tilde{\sigma}$. A **lift** of this ramification is a q cyclic Galois extension $L/K, \sigma$ which is unramified at C and has residue extension $\tilde{L}/F(C), \tilde{\sigma}$. Let P be a curve point on C, and $\pi_C \in R_P = \mathcal{O}_{S,P}$ a prime defining C at P. Note that the ramification data $\tilde{L}/F(C), \tilde{\sigma}$ can only ramify at nodes, and so is not ramified at P. Thus we can define $\bar{L}/F(P)$ to be the residue field extension, which has degree either q or 1. If δ is a prime of R_P, we say δ is a **split prime** at P if $F(\bar{\delta})$ contains $\bar{L}$.

Proposition 7.3. *Suppose δ is a split prime of R_P. If $\bar{L} \neq F(P)$, then δ has multiplicity of the form mq at P. Assume v is a valuation of K over P. Then either $v(\delta)$ is a multiple of q or the residue field of v contains $\bar{L}$.*

Proof. For the first part, let $\eta \in R_P$ be a prime nonsingular at P and assume $\bar{L} \neq F(P)$. By 0.3(d), the intersection multiplicity of $\delta = 0$ and $\eta = 0$ at P is a multiple of q. In particular, if η has tangent distinct from any tangent of $\delta = 0$, then δ has multiplicity a multiple of q. When the residue field $F(P)$ is finite, note that multiplicities are unchanged after etale extensions. Thus we can extend $F(P)$, with $\bar{L} \neq F(P)$ preserved, and be assured such an η exists.

As for the second part, we are done if $\bar{L} = F(P)$. If not, we can form the blow-up $X \to \mathrm{Spec}(R_P)$ with exceptional line E. Then v lies over E or a point of E. Let D be the curve in $\mathrm{Spec}(R_P)$ defined by δ. As a divisor on X, $(\delta) = mqE + D'$ where D' is the strict transform of D. If v lies over E, or a point on E not on D', it is clear that $v(\delta)$ is a multiple of q. Assume then that v lies over P' on E and D'. Set $R' = \mathcal{O}_{X,P'}$ with residue field $F(P')$. If $F(P')$ contains $\bar{L}$, then so does the residue field of v. If not, in R' we can write $\delta = y^{mq}\delta'$ where δ' defines D'. The integral closure of R'/δ' is a localization of $\bar{R}_\delta$ and so $F(\bar{\delta}')$ contains $\bar{L}$. We are done by induction on the blow-ups resolving $\delta = 0$. □

It turns out that we can use norms to create split primes. The following shows why we used the term "split."

Lemma 7.4. *Suppose L/K is a lift of $\tilde{L}/F(C)$ and P is a curve point of C. Assume L/K is not ramified at P. Let $R_P = \mathcal{O}_{S,P}$ and let T be the integral closure of R_P in L. Suppose $\delta \in R_P$ is a prime that splits in T. Then δ is a split prime.*

Proof. Since T/R_P is etale, T is regular. If $\bar{L} = F(P)$, there is nothing to prove, so we suppose otherwise. Since R_P is a UFD, $L = K(u^{1/q})$ where $u \in R_P^*$. Thus $\tilde{L} = F(C)(\tilde{u}^{1/q})$ where u maps to $\tilde{u}$. Let $\tilde{u}' \in R_P/\delta R_P$ be this other image of u. Since δ splits in L, $\tilde{u}'$ is a q power in $q(R_P/\delta R_P)$. It follows that $\tilde{u}'$ is a q power in $\bar{R}_\delta$ and hence in $F(\bar{\delta})$, as needed. □

We will use 7.4 to identify split primes. Note that δ splitting in L depends on our choice of lift L but our definition of split prime does not. Let $P \in C$ be a curve point and $\pi_C \in R_P = \mathcal{O}_{S,P}$ the prime defining C. We find split primes useful because:

Proposition 7.5. *Suppose $\pi' = \pi_C^s \Delta \in R_P$ where all the primes dividing Δ are split primes and s is prime to q. Then $K' = K(\pi'^{1/q})$ splits all the ramification of α over P.*

Proof. Just as in [S1, p. 32] we can write $\alpha = \alpha' + (u, \pi_C)_q$ where $u \in R_P^*$ and $\alpha' \in \mathrm{Br}(R_P)$. Thus, for some σ, $L = K(u^{1/q}), \sigma$ is a lift of the ramification data of α at C. Suppose v lies over P. Then α' is unramified with respect to v, $v(u) = 0$ and $v(\pi_C) > 0$. Thus the ramification of α with respect to v has the form $k(\bar{u}^{1/q}), \sigma'$ where $k \supset F(P)$ is the residue field of v. If k contains $\bar{L} = F(P)(\bar{u}^{1/q})$ or $v(\pi_C)$ is a multiple of q, then α is unramified at v. Otherwise, by 7.3, $v(\Delta)$ is divisible by q so $v(\pi')$ is prime to q and K'/K ramifies at v splitting the ramification of α by 0.1. □

We return to refining our choice of π, which so far has been chosen as in 7.2. The issue is to arrange to counteract the currently uncontrolled part of (π). Write $(\pi) = \sum_i s_i C_i + \sum_j t_j E_j + qE$ where all the t_j are prime to q and none of the E_j goes through a nodal point.

Proposition 7.6.

(a) *Let P be a curve point on C_i and $L/F(C_i), \sigma_i$ the ramification of α at C_i. If $L/F(C_i)$ is split at P, then α is itself unramified at all DVRs lying over P.*
(b) *Let P be as above where $\tilde{L}/F(C_i)$ is not split. Then the intersection multiplicity of $\sum_j t_j E_j$ and C_i at P is a multiple of q.*

Proof. If R_P is the stalk of S at P, there is a $\delta \in R_P$ with divisor $\sum_j t_j E_j$ where we have dropped terms not going through P. We can write $\alpha = \alpha' + (u, \pi_i)_q$ where $\alpha' \in \mathrm{Br}(R_p)$, π_i defines C_i at P, and u is a unit at P with image $\bar{u}$ in $F(P)$.

In (a), $\bar{u}$ is a q power. If v is a discrete valuation lying over P, then α' is unramified with respect to v and the residue field of v contains $F(P)$. Thus the ramification of α with respect to v is defined by some power of $\bar{u}^{1/q}$ and thus is trivial.

Turning to (b), since the residual class β_{C_i} is split it is certainly unramified at P. We are done after we prove:

Lemma 7.7. *Suppose P, C_i, $L/F(C_i)$, R_P, and π_i are as above and $\tilde{L}/F(C)$ is not split at P. In R_P suppose $\pi' = v\pi_C^s \delta' \eta^q$ where s is prime to q, $v \in R_P^*$ and δ', η are not divisible by π_C. Suppose $\beta' \in \mathrm{Br}(F(C))$ is the residual class of α with respect to $K' = K(\pi'^{1/q})$. Then β' is unramified at P if and only if the intersection multiplicity of $\delta' = 0$ and C at P is a multiple of q.*

Proof. It is important to recall we have assumed P is a nonsingular point on C. We can write $\alpha = \alpha' + (u, \pi_C)_q$ as above where now the image $\bar{u} \in F(P)^*$ is a non q power. Let $s's$ be congruent to 1 modulo q. Then, in $\mathrm{Br}(K')$, α has the same image as $\alpha' + (u^{s'}, v^{-1}\delta'^{-1})_q$ and β' is the image of this class in $\mathrm{Br}(F(C))$. The image of α' is unramified at P. If v_P is the valuation on $F(C)$ corresponding to P, the ramification of β' at P is given by an extension $k(u^{-s'v_P(\tilde{\delta}')/q})$ where $\tilde{\delta}'$ is the image of δ' in $F(C)$. This shows β' is unramified if and only if $v_P(\tilde{\delta}')$ is a q multiple. But by 0.3(c) this is the intersection multiplicity. □

By 7.6(a) we can ignore points P on the ramification locus where the ramification itself splits. In particular, it suffices to prove our result on a new S where any of these points are removed. Thus:

Corollary 7.8. *We can assume $\sum_j t_j E_j$ intersects all points of all the C_i with multiplicity a multiple of q.*

Choose a set of **marked** points consisting of all nodal points, all intersection points on a C_i and E_j, and at least one point on each C_i and E_j. If any of these curves has no codimension 2 points the argument below extends easily.

Proposition 7.9. *There is a $z \in K$ defined at all C_i such that z maps to q powers in $F(C_i)$ and $(z) = \sum_j t_j E_j + E'$ where the prime to q components of E' miss all the marked points.*

Proof. Let R be the localization (stalk) at all the marked points. Then R is a UFD by 0.3(a). $\sum_j t_j E_j$ is principal in R and so has the form (z). Let $I(C_i) \subset R$ be the ideal associated to the curve C_i. $R/I(C_i)$ is the stalk of the curve C_i at all the marked points on C_i. In particular, since C_i is nonsingular, $R/I(C_i)$ is a semilocal Dedekind domain (and hence a PID). It follows from 7.8 that if $\tilde{z}_i$ is the image of z in $R/I(C_i)$, then the divisor $(\tilde{z}_i) = (f_i^q)$ for some $f_i \in R/I(C_i)$. Thus $\tilde{z}_i = \tilde{v}_i f_i^q$ for $\tilde{v}_i$ a unit in $R/I(C_i)$. Note that f_i is a unit at all relevant nodal points since its divisor misses these.

Now $(I(C_i) + I(C_j))$ is the intersection of the maximal ideals corresponding to nodal points on both C_i and C_j. Since z has a unique image in each nodal point, the image of $\tilde{v}_i$ and $\tilde{v}_j$ must differ by a q power. Since $R/I(C_i)$ is semilocal, we can adjust the f_i by a unit and assume the $\tilde{v}_i$ have equal images at all nodal points. Thus by 0.3(b) there is a $v \in R^*$ a preimage of all the $\tilde{v}_i$. Adjusting z by v^{-1}, we are done. □

Next we consider creating lifted extensions L/K over a series of curves, and with further good properties. The first step is the following result. Recall we have assumed K has a primitive q root of one ρ and that we have blown up to eliminate chilly loops. Write $L_i = F(C_i)(u_i^{1/q})$.

Proposition 7.10.

(a) *There is a choice of u_i such that the following holds. Let P be a nodal point at the intersection of C_i and C_j. If P is a chilly point, then u_i and u_j have the same nonzero image in $F(P)$. If P is a cold point, then u_i and u_j are defined and hence 0 at P.*

(b) *There is a $u \in K$ defined on all C_i such that u maps to the u_i in* (a) *and the divisor (u) misses all marked points except cold nodal points.*

Proof. We begin with (a). Suppose P is a chilly nodal point. Then $L_i/F(C_i)$ are unramified at all such P. Choose the $u_i \in F(C_i)^*$ such that $\sigma_i(u_i^{1/q})/u_i^{1/q} = \rho$. Thus u_i has valuation a multiple of q at such P. At the cold points, the u_i have prime to q valuation. By weak approximation we can modify u_i by a q power, constrained at each nodal point, such that all the u_i are units at all the chilly points and have positive valuation at any cold points. Note that for any finite set of additional predesignated non-nodal points, we can also assume the u_i are units at those additional points (since $L_i/F(C_i)$ is unramified there also).

If s is the coefficient at P with respect to C_i, then u_j^s/u_i maps to a q power in $F(P)$. By 7.2 we can choose $s_i \in (\mathbb{Z}/q\mathbb{Z})^*$ for each C_i such that if $v_i = u_i^{s_i}$, then v_i/v_j maps to a q power in each $F(P)$. Again by weak approximation we can assume the v_i and v_j map to the same element in each $F(P)$, proving (a). As for (b), we proceed exactly as in 7.9. □

Corollary 7.11. *Suppose $(\pi) = \sum_i s_i C_i + \sum_j t_j E_j$. Then there is a $\delta \in K$ such that $K(\delta^{1/q})$ is a lifting of all the $L_i/F(C_i)$ and $(\delta) = \sum_j t_j E_j + E'$ where E' contains none of the C_i or E_j and E' misses all the non-cold marked points.*

Proof. Set $\delta = zu$, where z is from 7.9 and u is from 7.10. $\square$

Lemma 7.12. *There is an $x \in K$ such that $(x) = \sum_j t_j E_j + E''$ where the support of E'' does not contain any C_i, or any of the components of E', any of the intersection points of E' components and C_i's, or any intersection points of the C_i's. Furthermore, all the curves in E'' with prime to q coefficients in (x) are split at any intersection point with a C_i.*

Proof. Enlarge the set of marked points to include points on all components of E' and all intersection points of E' components and C_i's. Let R be the stalk of S at all these marked points and T the integral closure of R in $K(\delta^{1/q})$. Note that T is a normal semilocal ring, but need not be regular. Let $\mathcal{P}$ be the set of all the prime ideals of T extending prime ideals corresponding to C_i, E_j, and the components of E'. Note that the E_j extend uniquely because $K(\delta^{1/q})$ is ramified at those prime ideals.

We claim we can choose $w \in T$ which, for all j, has value t_j at the prime ideal extending E_j, is a unit at all other prime ideals in $\mathcal{P}$, and is a unit at all points over intersection points of among the C_i and between C_i's and E' components. To see this, let $P_j \subset T$ correspond to the extension of the E_j, $\{M_k\}$ be all the points in $\mathrm{Spec}(T)$ over nodal points, C_i and E' component intersection points, and intersection points among the E' components not on any E_j. Finally, let Q_l be all the primes over C_i and E' components which do not contain any of the M_k points. Since T is normal, its localization at a finite set of height one prime ideals is a UFD. Thus there is a $w' \in T$ with value t_j with respect to all the P_j and value 0 with respect to all the Q_l. Let I be the intersection of all $P_j^{t_j+1}$ and all Q_l. By construction, $I \cap M_1 \cap \cdots \cap M_{k-1} + M_k = T$, so by 0.2 and induction there is a w which maps to a nonzero element modulo any M_k and is congruent to w' modulo I. This w is as claimed.

Now let $N : K(\delta^{1/q}) \to K$ be the norm map and $x = N(w)$. Then x has valuation t_j with respect to E_j since $K(\delta^{1/q})$ is ramified there. If R_P is the stalk at any point the image of an M_k point, then the localization, T', of T at the points over P is the integral closure of R_P, and so $N(T'^*) \subset R_P^*$. In particular, x is a unit at all such P. Finally, let D be a curve in E''. Then E'' does not contain such P and is not among the C_i and components of E'. If $D \cap C_i$ does not contain any intersection point of a component of E' and C_i, or any nodal point. In particular, if P' is a point on $D \cap C_i$, then P' is a curve point where $K(\delta^{1/q})$ is unramified. Let R' be the stalk at P', and T' its integral closure in $K(\delta^{1/q})$. If D is unsplit in T, then the valuation of x at D must be a multiple of q. If D splits in T, then D is a split prime at P by 7.4.

Theorem 7.13. *$\pi' = x^{-1}\pi$ has the property that $K(\pi'^{1/q})$ splits all the ramification of α.*

Proof. We can write $(\pi') = \sum_i s_i C_i + E$ where any prime ideal in E either has coefficient a multiple of q, or is a split prime at any intersection point with a C_i. Furthermore, note from the proof of 7.12 that $x = N(w)$ where w is a unit at all extensions of the C_i in $K(\delta^{1/q})$. In particular, the image $\tilde{x}_i \in F(C_i)^*$ is a norm from L_i.

Let β'_{C_i} be the residual classes of α with respect to $K(\pi'^{1/q})$. By [S2] 1.8 the β'_{C_i} satisfy $\beta_{C_i} = \beta'_{C_i}\Delta(L_i/F(C_i), \sigma_i, \tilde{x}_i)$, so $\beta'_{C_i} = \beta_{C_i}$. This $K(\pi'^{1/q})$ split all the ramification of α at the C_i and over all nodal points.

It suffices then to consider curve points. If P is such a point on C_i, not in the support of E, we are done. If P only appears in components of E with q multiple coefficient, we are also done. Suppose then that $\sum_j t_j E'_j$ is the sum of components of (π') not among the C_i which contain P and with t_j prime to q. We are done by 7.5. $\square$

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